

CLASSIFICATION OF MAXIMAL TRANSITIVE PROLONGATIONS OF SUPER-POINCARÉ ALGEBRAS

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ABSTRACT. Let V be a complex orthogonal vector space and \mathbb{S} an irreducible $\mathcal{C}\ell(V)$ -module. A supertranslation algebra is a \mathbb{Z} -graded Lie superalgebra $\mathfrak{m} = \mathfrak{m}_{-2} + \mathfrak{m}_{-1} = V + (\mathbb{S} + \cdots + \mathbb{S})$ whose bracket $[\cdot, \cdot]_{\mathfrak{m}_{-1} \otimes \mathfrak{m}_{-1}}$ is $\mathfrak{so}(V)$ -invariant and non-degenerate. We consider the maximal transitive prolongations in the sense of Tanaka of supertranslation algebras. We prove that they are finite-dimensional for $\dim V \geq 3$ and classify them in terms of super-Poincaré algebras and appropriate \mathbb{Z} -gradations of simple Lie superalgebras.

1. INTRODUCTION

The theory of Lie superalgebras became a mainstream topic of research during the '70s, the interest being mainly motivated by the problem of constructing supersymmetric field theories, in particular supergravity [14, 27]. One of the first Lie superalgebras to be considered was the $d = 4$ super-Poincaré algebra, obtained from the isometry algebra $\mathfrak{isom}(\mathbb{C}^4) = \mathfrak{so}(4) + \mathbb{C}^4$ of the flat orthogonal space \mathbb{C}^4 by adding “odd spinorial generators”. More precisely, given any orthogonal vector space V of dimension $\dim V = d$, a complex Lie superalgebra $\mathfrak{p} = \mathfrak{p}_0 + \mathfrak{p}_1$ is called a *super-Poincaré algebra* if

- $\mathfrak{p}_0 = \mathfrak{so}(V) + V$ is the usual Poincaré algebra;
- $\mathfrak{p}_1 = \mathbb{S} + \cdots + \mathbb{S}$ is sum of N copies of an irreducible module \mathbb{S} for the Clifford algebra $\mathcal{C}\ell(V)$;
- $[V, \mathfrak{p}_1] = 0$;
- the action of $\mathfrak{so}(V)$ on \mathfrak{p}_1 is the spinor representation

$$\gamma : \mathfrak{so}(V) \rightarrow \mathfrak{gl}(\mathfrak{p}_1); \quad (1.1)$$

- the bracket between odd elements takes values in V and is given by a non-degenerate $\mathfrak{so}(V)$ -equivariant symmetric bilinear form

$$\Gamma : \mathfrak{p}_1 \vee \mathfrak{p}_1 \rightarrow V \quad (1.2)$$

satisfying an admissibility condition, see (2.1).

A complete classification of super-Poincaré algebras has been achieved in the '90s by Alekseevsky and Cortés: the main result of [1] is indeed an explicit description of a basis, consisting of admissible tensors, of the space of $\mathfrak{so}(V)$ -invariant elements in $\vee^2 \mathbb{S}^* \otimes V$.

Key words and phrases. Tanaka prolongations, super-Poincaré algebras, superconformal algebras, Clifford algebras and spinors.

The second author was supported by the project F1R-MTH-PUL-08HALO-HALOS08 of the University of Luxembourg.

It is easy to see that any super-Poincaré algebra

$$\mathfrak{p} = \mathfrak{p}_0 + \mathfrak{p}_1 = (\mathfrak{so}(V) + V) + (\mathbb{S} + \cdots + \mathbb{S}) \quad (1.3)$$

admits a natural realization as an algebra of super-vector fields

$$\begin{aligned} \frac{\partial}{\partial z^i}, & & V, \\ \frac{\partial}{\partial \xi^\alpha} - \frac{1}{2} \xi^\beta \Gamma_{\beta\alpha}^i \frac{\partial}{\partial z^i}, & & \mathbb{S} + \cdots + \mathbb{S}, \\ z_i \frac{\partial}{\partial z^j} - z_j \frac{\partial}{\partial z^i} + \gamma_{ij\alpha}^\beta \xi^\alpha \frac{\partial}{\partial \xi^\beta}, & & \mathfrak{so}(V), \end{aligned}$$

on the flat superspace $\mathbb{M} = \mathbb{C}^{d|(N\dim \mathbb{S})}$ with even and odd coordinates (z^i, ξ^α) . The super-Poincaré algebra (1.3) inherits a natural \mathbb{Z} -gradation

$$\mathfrak{p} = \mathfrak{p}_{-2} + \mathfrak{p}_{-1} + \mathfrak{p}_0 = V + (\mathbb{S} + \cdots + \mathbb{S}) + \mathfrak{so}(V)$$

by assigning degrees $\deg x^i = 2 = -\deg \frac{\partial}{\partial x^i}$, $\deg \xi^\alpha = 1 = -\deg \frac{\partial}{\partial \xi^\alpha}$; its negatively graded part

$$\mathfrak{m} = \mathfrak{m}_{-2} + \mathfrak{m}_{-1} = \mathfrak{p}_{-2} + \mathfrak{p}_{-1} = V + (\mathbb{S} + \cdots + \mathbb{S}) \quad (1.4)$$

is called a *supertranslation algebra* [10, 36]. The simply connected nilpotent Lie supergroup associated to (1.4) is clearly identifiable with the flat superspace \mathbb{M} of dimension $(d|N\dim \mathbb{S})$ and one can associate to $\mathfrak{m}_{-1} \subset T_e \mathbb{M}$ a unique \mathfrak{p} -invariant distribution \mathcal{D} on \mathbb{M} [33, 31]. It has depth 2 and the Levi form

$$\mathcal{L}_x : \mathcal{D}_x \vee \mathcal{D}_x \rightarrow \mathcal{T}_x \mathbb{M} / \mathcal{D}_x, \quad (X, Y) \mapsto [X, Y]_x \pmod{\mathcal{D}_x}$$

is identifiable with the tensor (1.2) at any point x of the body of \mathbb{M} . In particular, the non-degeneracy of (1.2) implies that \mathcal{D} is highly non-integrable.

In [32, 33, 3] the following notion of curved analogs of the pairs $(\mathbb{M}, \mathcal{D})$ has been considered.

Definition 1.1. Let $\mathfrak{p} = \mathfrak{m}_{-2} + \mathfrak{m}_{-1} + \mathfrak{p}_0 = V + (\mathbb{S} + \cdots + \mathbb{S}) + \mathfrak{so}(V)$ be the super-Poincaré algebra determined by a tensor (1.2). A *super-Poincaré structure* on a supermanifold M of dimension $\dim M = (\dim \mathfrak{m}_{-2} | \dim \mathfrak{m}_{-1})$ is a depth 2 distribution \mathcal{D} with $\text{rank } \mathcal{D} = \dim \mathfrak{m}_{-1}$ whose Levi form \mathcal{L}_x is identifiable with the tensor (1.2) at all points x of the body of M .

Our motivation to study super-Poincaré structures relies on the interesting fact that supergravity theories admit, besides the traditional “component formalism” formulations [14, 27], more geometric presentations in terms of super-Poincaré structures (M, \mathcal{D}) (see [25, 28, 29, 30, 32, 33]). The physical fields of the component formalism presentation, as well as the equations they satisfy, can be represented by appropriate tensorial objects on the supermanifold M and the supersymmetry transformations by Lie derivatives along sections of the distribution \mathcal{D} .

The main result of this work (Theorem 5.1) is the explicit description of the *maximal transitive prolongation* \mathfrak{g} in the sense of N. Tanaka [35] of a supertranslation algebra (1.4), for all possible dimensions d and all $N \in \mathbb{N}$. We recall here the definition: \mathfrak{g} is the unique \mathbb{Z} -graded Lie superalgebra $\mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \cdots$ which satisfies

- $\mathfrak{g}_p = \mathfrak{m}_p$ for $p = -1, -2$,
- for all $p \geq 0$, if $X \in \mathfrak{g}_p$ and $[X, \mathfrak{m}_{-1}] = 0$ then $X = 0$,
- \mathfrak{g} is maximal with these properties.

In geometrical terms \mathfrak{g} is the algebra of all formal infinitesimal symmetries of the homogeneous model $(\mathbb{M}, \mathcal{D})$.

If $\dim V = 1$, the maximal transitive prolongation \mathfrak{g} is infinite-dimensional and isomorphic to the contact Lie superalgebra $K(1, N)$ described in [22]; indeed a supermanifold of dimension $(1|N)$ endowed with a super-Poincaré structure is just a contact supermanifold. If $\dim V = 2$, then \mathfrak{g} is the sum of two copies of $K(1, N)$.

If $\dim V \geq 3$, the maximal transitive prolongation \mathfrak{g} is finite-dimensional. A similar statement has been proved in the Lie algebra case [3] by applying a deep result of N. Tanaka and J.-P. Serre [35, 17] (see Remark 2.7). In the Lie superalgebra case, the aforementioned result is not valid anymore; our proof of finite-dimensionality uses different techniques and relies on V. Kac's classification of infinite-dimensional simple linearly compact Lie superalgebras [22]. In all cases except those listed in Theorem 4.11, Table 3, the prolongation \mathfrak{g} satisfies $\mathfrak{g}_p = 0$ for all $p \geq 1$ or, equivalently, $\mathfrak{g} = \mathfrak{p} + \mathbb{C}E + \mathfrak{h}_0$ where

$$E = \xi^\alpha \frac{\partial}{\partial \xi^\alpha} + 2z^i \frac{\partial}{\partial z^i}$$

denotes the Euler vector field and $\mathfrak{h}_0 = \{D \in \mathfrak{g}_0 \mid [D, \mathfrak{m}_{-2}] = 0\}$ the algebra of internal symmetries of \mathfrak{m}_{-1} .

The cases in which the positively graded part of \mathfrak{g} is not trivial are listed in Table 3 and they are

- $\mathfrak{g} = osp(N|4)$, $\dim V = 3$ and \mathfrak{m} as described in Example 4.8,
- $\mathfrak{g} = pgl(4|N)$, $\dim V = 4$ and \mathfrak{m} as described in Example 4.7,
- $\mathfrak{g} = F(4)$, $\dim V = 5$, $N = 2$ and \mathfrak{m} as described in Example 4.9.

The Lie superalgebras $osp(N|4)$ and $sl(4|N)$ with the \mathbb{Z} -gradation described in Table 3 already appeared in the literature under the name of *superconformal algebras* [25]. The Lie superalgebra $\mathfrak{g} = osp(8|2m)$ is also a superconformal algebra with $\dim V = 6$; its negatively graded part is described in Example 4.10 and, although not isomorphic to a supertranslation algebra, admits a similar description in terms of semi-spinor modules \mathbb{S}^+ .

The existence of a non trivial positively graded part of \mathfrak{g} has a geometrical significance: it provides additional local symmetries of \mathbb{M} . In this case, the inclusion of \mathfrak{m} in \mathfrak{g} induces an open dense embedding of \mathbb{M} into the flag supermanifold $\overline{\mathbb{M}} = G/G_{\geq 0}$, where G denotes the simply connected Lie supergroup with Lie superalgebra \mathfrak{g} and $G_{\geq 0}$ the parabolic subsupergroup associated to $\sum_{i \geq 0} \mathfrak{g}_i$. We believe that this phenomenon is responsible for the off-shell nature of the supergravity theories in $\dim V \leq 6$ modeled on the Lie superalgebras of Examples 4.7, 4.8, 4.9, 4.10. Indeed a crucial step in super-space formulations of supergravity theories is the choice of a class of connections compatible with the distribution \mathcal{D} and satisfying appropriate torsion constraints [25, 28, 32, 33]; for $\dim V \leq 6$ the torsion constraints do not determine the connection uniquely. Deeper investigations on this subject will be the content of future work.

The paper is structured as follows. In Section 2 we give the relevant definitions and adapt to the Lie superalgebra case some of the results already proved in [3] for Lie algebras, giving in particular Theorem 2.3 on the structure of \mathfrak{g}_0 .

Section 3 specializes to the case $\dim V \geq 3$ and contains most of the technical results of the paper. Therein we first prove that the maximal transitive prolongation \mathfrak{g} either satisfies $\mathfrak{g}_p = 0$ for all $p \geq 1$, or is semisimple and contains a unique minimal ideal \mathfrak{s} , which is a simple prolongation of \mathfrak{m} (Theorem 3.1). Then, with the help of Kac's classification of \mathbb{Z} -graded even transitive irreducible infinite dimensional Lie superalgebras and strongly transitive modules [22], we show that \mathfrak{s} and \mathfrak{g} are finite-dimensional.

In Section 4 we first classify all simple Lie superalgebras \mathfrak{s} that arise as prolongations of a supertranslation algebra (Theorem 4.5). They are all *basic* Lie superalgebras (i.e. not of Cartan type or belonging to the *strange* series P and Q) and the \mathbb{Z} -gradations are given in terms of Dynkin diagrams. Theorem 4.11 then describes the maximal transitive prolongations \mathfrak{g} whose minimal ideal \mathfrak{s} is one of the Lie superalgebras of Theorem 4.5.

Section 5 contains the full classification result (Theorem 5.1) and the final Section 6 is devoted to the comparison of the results of this paper with the corresponding ones in the Lie algebra case [3].

Notation. For any supervector space $U = U_{\bar{0}} \oplus U_{\bar{1}}$, we denote by $\Pi U = (\Pi U)_{\bar{0}} \oplus (\Pi U)_{\bar{1}}$ the supervector space with opposite parity, that is

$$(\Pi U)_{\bar{0}} = U_{\bar{1}} \quad , \quad (\Pi U)_{\bar{1}} = U_{\bar{0}} \quad (1.5)$$

as (non-super) vector spaces and the tensor product $U \otimes U'$ between two supervector spaces is the supervector space

$$(U \otimes U')_{\bar{0}} = (U_{\bar{0}} \otimes U'_{\bar{0}}) \oplus (U_{\bar{1}} \otimes U'_{\bar{1}}) \quad , \quad (U \otimes U')_{\bar{1}} = (U_{\bar{0}} \otimes U'_{\bar{1}}) \oplus (U_{\bar{1}} \otimes U'_{\bar{0}}) \quad .$$

The skew-symmetric and symmetric tensors are defined as

$$\bigwedge U = \bigotimes U / \langle x \otimes y + y \otimes x \mid x, y \in U \rangle \quad , \quad \bigvee U = \bigotimes U / \langle x \otimes y - y \otimes x \mid x, y \in U \rangle$$

Finally, throughout the paper we adopt the convention that \mathbb{S} is a *purely odd* supervector space.

2. SUPERTRANSLATION ALGEBRAS AND THEIR PROLONGATIONS

2.1. Main definitions. Let V be a complex finite dimensional vector space, with a nondegenerate symmetric bilinear form (\cdot, \cdot) and let $\mathcal{Cl}(V) = \mathcal{Cl}(V)^{\bar{0}} + \mathcal{Cl}(V)^{\bar{1}}$ be the associated Clifford algebra with its natural \mathbb{Z}_2 -gradation. We adopt the conventions used in [23], in particular the product in $\mathcal{Cl}(V)$ satisfies $vu + uv = -2(v, u)\mathbf{1}$ for any $v, u \in V$.

We will denote by \mathbb{S} an irreducible $\mathcal{Cl}(V)$ -module. If $\dim V$ is odd, there exist two inequivalent irreducible $\mathcal{Cl}(V)$ -modules; they are equivalent irreducible $\mathfrak{so}(V)$ -modules. If $\dim V$ is even, there exists only one irreducible $\mathcal{Cl}(V)$ -module \mathbb{S} . It is $\mathfrak{so}(V)$ -reducible and we denote by \mathbb{S}^+ and \mathbb{S}^- its $\mathfrak{so}(V)$ -irreducible components. We call \mathbb{S} the *spinor* representation of $\mathfrak{so}(V)$ and \mathbb{S}^\pm the *semi-spinor* representations, when $\dim V$ is even.

Remark 2.1. The reader should not confuse *semi-spinor* with *half-spin* representations. The latter are the finite-dimensional representations of $\mathfrak{so}(V)$ which integrate to $\text{Spin}(V)$ but do not integrate to $\text{SO}(V)$. Half-spin representations are characterized as sums of $\mathfrak{so}(V)$ -submodules of $\mathbb{S}^{\otimes 2n+1}$, $n \in \mathbb{N}$.

Let W be a $\mathcal{C}\ell(V)$ -module. The action of an element $c \in \mathcal{C}\ell(V)$ on $s \in W$ will be denoted by $c \cdot s$. We will denote by $N \geq 1$ the number of irreducible $\mathcal{C}\ell(V)$ -components of W , counted with their multiplicities (we note that this convention is not universally adopted, and some authors use “ N ” to denote the number of irreducible $\mathfrak{so}(V)$ -components). Following [1], a nondegenerate bilinear form $\beta: W \times W \rightarrow \mathbb{C}$ is called *admissible* if there exist $\tau, \sigma \in \{\pm 1\}$ such that $\beta(v \cdot s, t) = \tau \beta(s, v \cdot t) = \sigma \beta(t, v \cdot s)$ for all $v \in V$ and $s, t \in W$. In [1], it is proved that the space of admissible bilinear forms on W is always non-trivial.

Consider an admissible bilinear form β on W such that $\sigma\tau = 1$. It satisfies the following properties:

- (B1) β is $\mathfrak{so}(V)$ -invariant,
- (B2) β is symmetric or skew-symmetric (we let $\epsilon = 1$ in the former case and $\epsilon = -1$ in the latter),
- (B3) for all $v \in V$ and $s, t \in W$, $\beta(v \cdot s, t) = \epsilon \beta(s, v \cdot t)$.

Set $\mathfrak{m}_0 = \mathfrak{m}_{-2} = V$, $\mathfrak{m}_1 = \mathfrak{m}_{-1} = W$, $\mathfrak{m} = \mathfrak{m}_{-2} + \mathfrak{m}_{-1}$. On \mathfrak{m} we define a structure of \mathbb{Z} -graded Lie superalgebra with bracket given, for $s, t \in W$ and $v \in V$, by

$$([s, t], v) = \beta(v \cdot s, t). \quad (2.1)$$

Note that the \mathbb{Z}_2 -gradation of \mathfrak{m} concides with the \mathbb{Z} -grading (mod 2). Gradations with this property will be called *consistent*.

Definition 2.2. Any consistent \mathbb{Z} -graded Lie superalgebra $\mathfrak{m} = \mathfrak{m}(V, W, \beta)$ with bracket defined as in (2.1) is called a *supertranslation algebra*.

We recall that the *maximal transitive prolongation* of \mathfrak{m} is a (possibly infinite-dimensional) \mathbb{Z} -graded Lie superalgebra $\mathfrak{g} = \sum_{p \in \mathbb{Z}} \mathfrak{g}_p$ such that:

- (1) \mathfrak{g}_p is finite dimensional for every $p \in \mathbb{Z}$;
- (2) $\mathfrak{g}_p = \mathfrak{m}_p$ for $p = -1, -2$ and $\mathfrak{g}_p = 0$ for $p < -2$;
- (3) for all $p \geq 0$, if $X \in \mathfrak{g}_p$ is an element such that $[X, \mathfrak{g}_{-1}] = 0$, then $X = 0$ (*transitivity*);
- (4) \mathfrak{g} is maximal with these properties, i.e. if \mathfrak{g}' is another \mathbb{Z} -graded Lie superalgebra satisfying (1), (2), and (3), then there exists an injective homomorphism of \mathbb{Z} -graded Lie superalgebras $\phi: \mathfrak{g}' \rightarrow \mathfrak{g}$.

The existence and uniqueness of \mathfrak{g} is proved in [35] (the proof is given in the Lie algebra case but it extends verbatim to the superalgebra case). Note that, by transivity, the maximal transitive prolongation of a consistently graded Lie superalgebra \mathfrak{m} is also consistently graded, that is $\mathfrak{g}_0 = \sum_p \mathfrak{g}_{2p}$ and $\mathfrak{g}_1 = \sum_p \mathfrak{g}_{2p+1}$.

From now on, we will identify without explicit mention the spaces \mathfrak{m}_{-1} , \mathfrak{g}_{-1} and W (resp. \mathfrak{m}_{-2} , \mathfrak{g}_{-2} and V).

2.2. Preliminary results. In this section we adapt to the Lie superalgebra case some of the results already proved in [3] for Lie algebras: we explicitly describe the subalgebra \mathfrak{g}_0 of the maximal transitive prolongation $\mathfrak{g} = \sum_{p \in \mathbb{Z}} \mathfrak{g}_p$ of a supertranslation algebra $\mathfrak{m} = V + W$, construct an equivariant embedding of \mathfrak{g}_1 into \mathfrak{g}_{-1} , and prove that the action of the positively graded part of \mathfrak{g} on \mathfrak{g}_{-2} is faithful. The results of this section will be frequently used, even without explicit mention, throughout the paper.

We recall that \mathfrak{g}_0 is the Lie algebra of 0-degree derivations of \mathfrak{m} . We will use indifferently the notations $[D, X]$ and DX for the bracket in \mathfrak{g} of an element $D \in \mathfrak{g}_0$ and an element $X \in \mathfrak{m}$.

Let now $E \in \mathfrak{g}_0$ be the derivation acting with eigenvalues -1 on \mathfrak{m}_{-1} and -2 on \mathfrak{m}_{-2} . We call E the *grading element* of \mathfrak{g} . Moreover, let \mathfrak{h}_0 be the set of elements in \mathfrak{g}_0 acting trivially on \mathfrak{m}_{-2} :

$$\mathfrak{h}_0 = \{D \in \mathfrak{g}_0 \mid [D, v] = 0 \ \forall v \in V\}.$$

We quote now some results from [3], whose proofs carry over unchanged to the Lie superalgebra case.

Theorem 2.3 ([3, Theorem 2.3]). *The Lie algebra \mathfrak{g}_0 is a direct sum of ideals:*

$$\mathfrak{g}_0 = \mathfrak{so}(V) \oplus \mathbb{C}E \oplus \mathfrak{h}_0$$

where $\mathfrak{so}(V)$ acts on \mathfrak{m}_{-1} and \mathfrak{m}_{-2} via the spin and the standard representations, respectively. \square

Lemma 2.4 ([3, Lemma 2.5]). *There exists a unique $\mathfrak{so}(V)$ -equivariant linear map $\phi: \mathfrak{g}_1 \rightarrow W$ satisfying*

$$Dv = v \cdot \phi(D)$$

for all $D \in \mathfrak{g}_1$ and $v \in V$. \square

Proposition 2.5 ([3, Proposition 2.6]). *For every $v \in V$, there exists a 0-degree Lie superalgebra homomorphism $\psi_v: \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying*

- (1) $\psi_v(s) = v \cdot s$ for all $s \in W$;
- (2) for all $u \in V$, $\psi_v(u) = \begin{cases} \epsilon(v, v) \left(u - \frac{2(v, u)}{(v, v)} v \right) & \text{if } v \text{ is non-isotropic,} \\ -2\epsilon(v, u)v & \text{if } v \text{ is isotropic;} \end{cases}$
- (3) $\psi_v(\phi(\psi_v(D))) = \epsilon\phi(D)$ for all $D \in \mathfrak{g}_1$.

Moreover ψ_v is invertible if and only if v is non-isotropic. \square

The next result corresponds to [3, Theorem 2.4 (2)]. There are however some differences between the classical and super case and it is appropriate to give a full proof.

Proposition 2.6. *Let $\mathfrak{g} = \sum_{p \in \mathbb{Z}} \mathfrak{g}_p$ be the maximal transitive prolongation of a supertranslation algebra $\mathfrak{m} = V + W$ with $\dim V \geq 3$. For all $p \geq 1$, if $X \in \mathfrak{g}_p$ and $[X, \mathfrak{g}_{-2}] = 0$ then $X = 0$.*

Proof. If $p \geq 0$ then $\{D \in \mathfrak{g}_p \mid [D, \mathfrak{g}_{-2}] = 0\} \simeq \mathfrak{h}_0^{(p)}$ where

$$\mathfrak{h}_0^{(p)} = (W \otimes \Lambda^{p+1} W^*) \cap (\mathfrak{h}_0 \otimes \Lambda^p W^*)$$

is the p -th Cartan superprolongation (see *e.g.* [15] for more details) of the even Lie superalgebra $\mathfrak{h}_0 \subset \mathfrak{gl}(W)$ acting on the purely odd superspace W .

Let $x, y, z \in V$ be orthogonal non-isotropic vectors and consider the bilinear form α on W defined by

$$\alpha(s, t) = ([y \cdot z \cdot s, t], x) = \beta(x \cdot y \cdot z \cdot s, t) . \quad (2.2)$$

Straightforward computations show that

- (1) α is skew-symmetric and nondegenerate,
- (2) $\mathfrak{h}_0 \subset \mathfrak{osp}(W, \alpha)$.

Since $\mathfrak{osp}(W, \alpha)^{(1)} = 0$ (see *e.g.* [15, Theorem 5.1]), also \mathfrak{h}_0 has a trivial Cartan superprolongation. \square

For $p = 1$, Proposition 2.6 asserts that the $\mathfrak{so}(V)$ -equivariant map $\phi: \mathfrak{g}_1 \rightarrow W$ of Lemma 2.4 is injective, i.e. any $D \in \mathfrak{g}_1$ is uniquely determined by its action on V . Moreover, by (3) of Proposition 2.5, the image $\phi(\mathfrak{g}_1)$ is a $\mathcal{Cl}(V)$ -submodule of W .

Remark 2.7. In the Lie algebra case, [3, Theorem 2.4] states also that the maximal transitive prolongation of an extended-translation Lie algebra $\mathfrak{m} = V + W$ with $\dim V \geq 3$ is *finite-dimensional*. This is a consequence of a deep theorem of Tanaka [35, Theorem 11.1] which is based on some arguments of Serre [17] on Spencer cohomology of Lie algebras. This theorem says that *the maximal transitive prolongation \mathfrak{g} of a fundamental Lie algebra $\mathfrak{m} = \mathfrak{m}_{-2} + \mathfrak{m}_{-1}$ is finite-dimensional if and only if the Cartan prolongation of the Lie algebra $\mathfrak{h}_0 = \{D \in \mathfrak{g}_0 \mid [D, \mathfrak{g}_{-2}] = 0\} \subset \mathfrak{gl}(\mathfrak{g}_{-1})$ is finite-dimensional.*

The naive generalization of Tanaka's result is not true for \mathbb{Z} -graded Lie superalgebras. As a counterexample, consider the infinite-dimensional exceptional semisimple Lie superalgebra $\mathfrak{g} = E'(5, 10)$ described in [9, §4.3]. It is the maximal transitive prolongation of the consistently graded Lie superalgebra $\mathfrak{m} = \mathfrak{m}_{-2} + \mathfrak{m}_{-1} = \mathbb{C}^{5*} + \Pi \Lambda^2 \mathbb{C}^5$ with bracket given by $[\alpha, \beta] = \iota_{\alpha \wedge \beta} \text{vol}$, for $\alpha, \beta \in \mathfrak{m}_{-1}$. The subalgebra \mathfrak{g}_0 is $\mathfrak{gl}(5, \mathbb{C})$ acting in the obvious way on \mathfrak{m} ; in particular $\mathfrak{h}_0 = 0$.

We will prove in section 3 that a transitive prolongation \mathfrak{g} of a super-translation algebra \mathfrak{m} with $\dim V \geq 3$ is finite-dimensional. Our proof does not rely on a generalization of Tanaka's result but rather on the existence (when $\mathfrak{g}_1 \neq 0$) of a “large” simple ideal $\mathfrak{m} \subset \mathfrak{s} \subset \mathfrak{g}$ (Theorem 3.1) and on the classification of \mathbb{Z} -graded even transitive irreducible infinite-dimensional Lie superalgebras given in [22].

In the low dimensional cases $\dim V = 1, 2$, the Lie superalgebra \mathfrak{g} is infinite-dimensional. These cases will be discussed in detail in section 5.

From now on, we will assume that $\dim V \geq 3$.

3. SEMISIMPLICITY AND FINITE-DIMENSIONALITY

3.1. Semisimplicity of the maximal prolongation. Recall that a (possibly infinite-dimensional) Lie superalgebra is called *semisimple* if it does not contain any non-zero abelian ideal. The main goal of this section is to prove the following theorem.

Theorem 3.1. *Let $\dim V \geq 3$ and $\mathfrak{g} = \sum_{p \in \mathbb{Z}} \mathfrak{g}_p$ be the maximal transitive prolongation of a supertranslation algebra $\mathfrak{m} = V + W$. Then exactly one of the following two cases occurs:*

- (1) $\mathfrak{g}_p = 0$ for all $p \geq 1$;
- (2) \mathfrak{g} is semisimple and contains a unique minimal non-zero ideal \mathfrak{s} .

In the latter case \mathfrak{s} is a simple transitive prolongation of \mathfrak{m} which contains $\sum_{p \geq 0} \mathfrak{g}_{2p+1}$ and the ideal $\mathfrak{so}(V)$ of \mathfrak{g}_0 described in Theorem 2.3.

Remark 3.2. Theorem 3.1 is also valid in the Lie algebra case, with essentially the same proof. It is then easy to show that $\mathfrak{g} = \mathfrak{s}$ is a simple Lie algebra in case (2). This improves [3, Theorem 2.7], extending the classification results obtained in [3] to arbitrary $N \geq 1$.

The proof of Theorem 3.1 requires an intermediate result.

Proposition 3.3. *Let \mathfrak{g} be the maximal transitive prolongation of a supertranslation algebra $\mathfrak{m} = V + W$ with $\dim V \geq 3$. If $\mathfrak{g}_1 \neq 0$, then any transitive prolongation \mathfrak{q} of \mathfrak{m} such that*

- (1) $\mathfrak{q}_1 \neq 0$,
- (2) \mathfrak{q}_0 contains the ideal $\mathfrak{so}(V) \oplus \mathbb{C}E$ described in Theorem 2.3,
- (3) \mathfrak{q} is preserved by any 0-degree Lie superalgebra automorphism of \mathfrak{g} ,

is a semisimple Lie superalgebra.

Proof. Let \mathfrak{k} be a non-zero ideal of \mathfrak{q} . Then \mathfrak{k} is \mathbb{Z} -graded since \mathfrak{q} contains the grading element E . If $\mathfrak{k}_{-2} = 0$ then, by non-degeneracy of β , one has $\mathfrak{k}_{-1} = 0$ and, by transitivity, $\mathfrak{k}_p = 0$ for every $p \geq 0$. It follows that every non-zero ideal of \mathfrak{q} contains $\mathfrak{q}_{-2} = V$, since $\mathfrak{so}(V)$ acts irreducibly on V .

Hence, there exists a unique minimal ideal \mathfrak{s} of \mathfrak{q} ; it is \mathbb{Z} -graded, $\mathfrak{s}_{-2} = V$, and $\mathfrak{s}_{-1} \neq 0$ because $\mathfrak{s}_{-1} \supset [\mathfrak{q}_1, V]$ and $[\mathfrak{q}_1, V] \neq 0$ by Proposition 2.6. Moreover \mathfrak{s} is preserved by any automorphism of \mathfrak{q} and, by hypothesis (3), also by any 0-degree automorphism of \mathfrak{g} . In particular \mathfrak{s}_{-1} is preserved by all homomorphisms $\{\psi_v\}_{v \in V}$ described in Proposition 2.5 and is a $\mathcal{Cl}(V)$ -submodule of W .

Assume by contradiction that \mathfrak{s} is abelian. It follows that \mathfrak{s}_{-1} is a non-zero β -isotropic $\mathcal{Cl}(V)$ -submodule of W and

$$\mathfrak{s}_{-1}^\perp = \{t \in \mathfrak{g}_{-1} \mid \beta(t, \mathfrak{s}_{-1}) = 0\} = \{t \in \mathfrak{g}_{-1} \mid [t, \mathfrak{s}_{-1}] = 0\}$$

is a proper $\mathcal{Cl}(V)$ -submodule of W containing \mathfrak{s}_{-1} . Denote by \mathfrak{a} a $\mathcal{Cl}(V)$ -submodule of W complementary to \mathfrak{s}_{-1}^\perp . As the bilinear form $\eta = \beta|_{\mathfrak{s}_{-1} \otimes \mathfrak{a}}$ is nondegenerate, one has the following decomposition of W into $\mathcal{Cl}(V)$ -submodules

$$W = \mathfrak{a} \oplus \mathfrak{s}_{-1}^\perp = \mathfrak{a} \oplus (\mathfrak{b} \oplus \mathfrak{s}_{-1}), \quad (3.1)$$

where \mathfrak{b} is the β -orthogonal complement to $\mathfrak{a} \oplus \mathfrak{s}_{-1}$ in W . The bilinear form β can be written in block-matrix form w.r.t. (3.1) as

$$\beta = \begin{pmatrix} \tilde{\eta} & 0 & \epsilon^T \eta \\ 0 & \hat{\eta} & 0 \\ \eta & 0 & 0 \end{pmatrix}, \quad \begin{aligned} \eta &= \beta|_{\mathfrak{s}_{-1} \otimes \mathfrak{a}}, \\ \tilde{\eta} &= \beta|_{\mathfrak{a} \otimes \mathfrak{a}}, \\ \hat{\eta} &= \beta|_{\mathfrak{b} \otimes \mathfrak{b}}. \end{aligned} \quad (3.2)$$

The key point is now to show that one can always choose a $\mathcal{C}\ell(V)$ -submodule \mathfrak{a} so that $\tilde{\eta} = 0$. We will denote by

$$\eta^* : \mathfrak{s}_{-1} \rightarrow \mathfrak{a}^*, \quad \tilde{\eta}^* : \mathfrak{a} \rightarrow \mathfrak{a}^*, \quad \hat{\eta}^* : \mathfrak{b} \rightarrow \mathfrak{b}^*$$

the linear maps induced by (3.2) and by $\psi_v^t : \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ the dual map of the homomorphism $\psi_v : \mathfrak{g} \rightarrow \mathfrak{g}$. It is immediate to check that

$$\eta^* \circ \psi_v = \epsilon \psi_v^t \circ \eta^*, \quad \tilde{\eta}^* \circ \psi_v = \epsilon \psi_v^t \circ \tilde{\eta}^*, \quad \hat{\eta}^* \circ \psi_v = \epsilon \psi_v^t \circ \hat{\eta}^*$$

for all $v \in V$; it follows that $\varphi = (\eta^*)^{-1} \circ \tilde{\eta}^* : \mathfrak{a} \rightarrow \mathfrak{s}_{-1}$ is $\mathcal{C}\ell(V)$ -equivariant. The $\mathcal{C}\ell(V)$ -submodule $\{2a - \varphi(a) | a \in \mathfrak{a}\}$ is then β -isotropic and complementary to \mathfrak{s}_{-1}^\perp . Without loss of generality, we may hence assume that $\tilde{\eta} = 0$, *i.e.* $[\mathfrak{a}, \mathfrak{a}] = 0$.

Define now a 0-degree Lie superalgebra automorphism $\chi : \mathfrak{g} \rightarrow \mathfrak{g}$ by

$$\chi|_{\mathfrak{s}_{-1}} = (\Phi^*)^{-1} \circ \eta^*, \quad \chi|_{\mathfrak{a}} = (\eta^*)^{-1} \circ \Phi^*, \quad \chi|_{\mathfrak{b}} = \sqrt{\mu} \text{Id}_{\mathfrak{b}}, \quad \chi|_{\mathfrak{g}_{-2}} = \mu \text{Id}_V$$

where $\mu = \pm 1$ and Φ is an admissible bilinear form on \mathfrak{a} . Since $\chi(\mathfrak{s}_{-1}) = \mathfrak{a}$, invariance of \mathfrak{s} under any 0-degree automorphism of \mathfrak{g} gives a contradiction. \square

Proof of Theorem 3.1. Assume that $\mathfrak{g}_1 \neq 0$. By Proposition 3.3, \mathfrak{g} is a semisimple Lie superalgebra. Arguing as in the first part of the proof of Proposition 3.3, one can prove that every ideal of \mathfrak{g} is \mathbb{Z} -graded and contains \mathfrak{g}_{-2} . The unique minimal non-zero ideal \mathfrak{s} of \mathfrak{g} is then the ideal generated by \mathfrak{g}_{-2} and it has a nonzero component \mathfrak{s}_{-1} in degree -1 .

Claim I. $\mathfrak{s}_0 \supset \mathfrak{so}(V)$.

If \mathfrak{s}_0 does not contain $\mathfrak{so}(V)$ then, by $\mathfrak{so}(V)$ -invariance (for $\dim V \neq 4$; by invariance under $\mathfrak{so}(V)$ and all automorphisms $\{\psi_v\}_{v \in V}$ if $\dim V = 4$), one gets $\mathfrak{s}_0 \subset \mathbb{C}E + \mathfrak{h}_0$. Consider an element $D \in \mathfrak{s}_1$. Then, for all $t \in \mathfrak{g}_{-1}$, the element Dt belongs to $\mathbb{C}E + \mathfrak{h}_0$ and

$$\begin{aligned} 0 &= ([Dt, v], w) - ([Dt, w], v) = ([t, Dv], w) - ([t, Dw], v) \\ &= \beta(w \cdot t, v \cdot \phi(D)) - \beta(v \cdot t, w \cdot \phi(D)) \\ &= \epsilon \beta((vw - wv) \cdot t, \phi(D)), \end{aligned} \tag{3.3}$$

where $\phi : \mathfrak{g}_1 \rightarrow W$ is the embedding in Lemma 2.4. It follows that $\phi(D) = 0$, hence $\mathfrak{s}_1 = 0$ and $\mathfrak{s}_p = 0$ for all $p \geq 1$ by transitivity.

As \mathfrak{s} is minimal and not abelian (by semisimplicity of \mathfrak{g}), we have $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}$. In particular $[\mathfrak{s}_0, \mathfrak{s}_0] = \mathfrak{s}_0$ is contained in $[\mathbb{C}E + \mathfrak{h}_0, \mathbb{C}E + \mathfrak{h}_0] \subset \mathfrak{h}_0$ and the center $Z(\mathfrak{s})$ of \mathfrak{s} is an ideal in \mathfrak{g} containing \mathfrak{g}_{-2} and strictly contained in \mathfrak{s} . This contradicts the minimality of \mathfrak{s} .

Claim II. $\mathfrak{s}_p = \mathfrak{g}_p$ for $p = -2, 2$ and for all odd p . Moreover $\mathfrak{s} = \mathfrak{g}_{\bar{1}} + [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$.

First note that \mathfrak{s} contains all nontrivial irreducible $\mathfrak{so}(V)$ -submodules in \mathfrak{g} because \mathfrak{s} is an ideal containing $\mathfrak{so}(V)$. For $p \geq 0$, we can identify \mathfrak{g}_p to an $\mathfrak{so}(V)$ -submodule of $W \otimes \bigotimes_{i=1}^{p+1} W^* \simeq \bigotimes_{i=1}^{p+2} W$, hence \mathfrak{g}_p is a sum of half-spin representations whenever p is odd. It follows that $\mathfrak{g}_p \subset \mathfrak{s}$ for p odd.

Proposition 2.6 implies that any trivial irreducible $\mathfrak{so}(V)$ -submodule of \mathfrak{g}_2 can be identified to a subspace $\mathbb{C}D$ of

$$\text{Hom}(V, \mathfrak{g}_0)^{\mathfrak{so}(V)} = \text{Hom}(V, \Lambda^2 V + \mathbb{C}^r)^{\mathfrak{so}(V)} = \text{Hom}(V, \Lambda^2 V)^{\mathfrak{so}(V)}.$$

The subspace $\mathfrak{g}_{-2} + \mathfrak{so}(V) + \mathbb{C}D$ of \mathfrak{g}_0 is a Lie subalgebra and D an element of the first Cartan prolongation of $\mathfrak{so}(V)$. It follows that $D = 0$ and $\mathfrak{g}_2 \subset \mathfrak{s}$.

Finally, $\mathfrak{g}_{\bar{1}} + [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ is an ideal of \mathfrak{g} contained in \mathfrak{s} , and so it is equal to \mathfrak{s} .

Claim III. \mathfrak{s} is simple.

Let \mathfrak{b} be a non-zero ideal of \mathfrak{s} and X a non-zero element of \mathfrak{b} . Then $X = \sum_i X_i$ is a finite sum of homogeneous elements $X_i \in \mathfrak{s}_i$ and we denote by j the highest integer for which $X_j \neq 0$. By transitivity and nondegeneracy of \mathfrak{g} , there exist elements $s_1, \dots, s_{j+2} \in \mathfrak{g}_{-1} = \mathfrak{s}_{-1}$ such that

$$0 \neq [s_1, \dots, [s_{j+2}, X] \dots] = [s_1, \dots, [s_{j+2}, X_j] \dots] \in \mathfrak{b} \cap \mathfrak{g}_{-2}.$$

It follows that $\mathfrak{g}_{-2} \subset \mathfrak{b}$. The ideal \mathfrak{a} generated by \mathfrak{g}_{-2} is the minimal non-zero ideal of \mathfrak{s} and it is \mathbb{Z} -graded. By Proposition 2.6 and transitivity we have that $\mathfrak{a}_{-1} \neq 0$.

If \mathfrak{a}_0 has a nontrivial $\mathfrak{so}(V)$ component then, arguing as in Claim I, $\mathfrak{a}_0 \supset \mathfrak{so}(V)$. Hence, as in Claim II, $\mathfrak{a}_{\bar{1}} = \mathfrak{s}_{\bar{1}} = \mathfrak{g}_{\bar{1}}$ and $\mathfrak{a} = \mathfrak{g}_{\bar{1}} + [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \mathfrak{s}$.

Otherwise, $\mathfrak{a}_0 \subset \mathbb{C}E + \mathfrak{h}_0$ and, arguing as in (3.3), we get $\mathfrak{a}_p = 0$ for all $p \geq 1$. If $[\mathfrak{a}, \mathfrak{a}] \neq 0$ then $\mathfrak{a}_0 \subset \mathfrak{h}_0$ and the center $Z(\mathfrak{a})$ of \mathfrak{a} is a non-zero abelian ideal of \mathfrak{s} . If $[\mathfrak{a}, \mathfrak{a}] = 0$ then \mathfrak{a} is abelian.

In both cases we find a non-zero \mathbb{Z} -graded abelian ideal of \mathfrak{s} , and hence of $\mathfrak{s} + \mathbb{C}E$. This gives a contradiction, as $\mathfrak{s} + \mathbb{C}E$ is semisimple by Proposition 3.3. \square

3.2. Finite dimension of the maximal prolongation. In this section we prove the following.

Theorem 3.4. *The maximal transitive prolongation \mathfrak{g} of a supertranslation algebra $\mathfrak{m} = V + W$ with $\dim V \geq 3$ is finite-dimensional.*

Before proving Theorem 3.4, we briefly recall some important notions about infinite-dimensional filtered Lie superalgebras. Every \mathbb{Z} -graded Lie superalgebra $\mathfrak{a} = \sum_{p \geq -2} \mathfrak{a}_p$ has a natural filtration:

$$\mathfrak{a} = \mathfrak{a}_{(-2)} \supset \mathfrak{a}_{(-1)} \supset \dots \supset \mathfrak{a}_{(p)} \supset \dots$$

where $\mathfrak{a}_{(p)} = \sum_{i \geq p} \mathfrak{a}_i$. On the direct product $\bar{\mathfrak{a}} = \prod_p \mathfrak{a}_p$ we consider the linear topology for which $\{\bar{\mathfrak{a}}_{(p)} = \prod_{i \geq p} \mathfrak{a}_i\}_{p \geq 2}$ is a fundamental system of finite-codimensional neighborhoods of 0. Then $\bar{\mathfrak{a}}$ is a complete topological algebra and it is *linearly compact* according to the definition in [16, 22]. A subspace of $\bar{\mathfrak{a}}$ is open if and only if it is closed and of finite codimension. There are a natural dense inclusion $\mathfrak{a} \subset \bar{\mathfrak{a}}$ and a natural isomorphism $\mathfrak{a} \simeq \text{gr}(\bar{\mathfrak{a}})$ where $\text{gr}(\bar{\mathfrak{a}})$ is the graded Lie superalgebra associated to the filtration of $\bar{\mathfrak{a}}$.

A proper open subalgebra $\mathcal{L}_0 \subset \bar{\mathfrak{a}}$ that does not contain any non-zero ideal of $\bar{\mathfrak{a}}$ is called *fundamental*. The linearly compact Lie superalgebra $\bar{\mathfrak{a}}$ admits a fundamental subalgebra if and only if it satisfies an artinian condition: every descending sequence of closed ideals of $\bar{\mathfrak{a}}$ is eventually stable [16, 4]. A maximal subalgebra $\mathcal{L}_0 \subset \bar{\mathfrak{a}}$ that is also fundamental is called *primitive*.

There exists also a stronger notion of primitivity. An even element $X \in \bar{\mathfrak{a}}$ is called *exponentiable* if $\text{ad}_{\bar{\mathfrak{a}}}(X)$ leaves invariant any closed subspace $H \subset \bar{\mathfrak{a}}$ that is invariant for every continuous automorphism of $\bar{\mathfrak{a}}$ [16, 5]. It is

known that every even element of a fundamental subalgebra is exponentiable [16, 22]. A primitive subalgebra $\mathcal{L}_0 \subset \bar{\mathfrak{a}}$ which contains *all* exponentiable elements of $\bar{\mathfrak{a}}$ is called *even primitive*.

The following intermediate result will be used in the proof of Theorem 3.4.

Proposition 3.5. *Let \mathfrak{s} be a simple transitive prolongation of a supertranslation algebra $\mathfrak{m} = V + W$ with $\dim V \geq 3$. If*

- $\mathfrak{so}(V) \subset \mathfrak{s}_0$,
- \mathfrak{s} is invariant under all 0-degree automorphisms of the maximal transitive prolongation \mathfrak{g} of \mathfrak{m} ,

then the completion $\bar{\mathfrak{s}}$

- (1) *is simple (i.e. it does not contain any proper non-zero closed ideal),*
- (2) *admits a primitive subalgebra \mathcal{L}_0 containing $\bar{\mathfrak{s}}_{(0)}$.*

If \mathfrak{s} is infinite-dimensional, any primitive subalgebra \mathcal{L}_0 containing $\bar{\mathfrak{s}}_{(0)}$ is even primitive and satisfies $\bar{\mathfrak{s}}_{(0)} \subset \mathcal{L}_0 \subsetneq \bar{\mathfrak{s}}_{(-1)}$.

Proof. Let \mathfrak{b} be a proper non-zero closed ideal of $\bar{\mathfrak{s}}$. The associated \mathbb{Z} -graded subalgebra $\text{gr}(\mathfrak{b})$ of $\text{gr}(\bar{\mathfrak{s}}) \simeq \mathfrak{s}$ is a proper non-zero ideal, proving (1).

Any maximal proper subalgebra \mathcal{L}_0 of $\bar{\mathfrak{s}}$ containing $\bar{\mathfrak{s}}_{(0)}$ is open and fundamental, proving (2).

Assume now that $\bar{\mathfrak{s}}$ is infinite-dimensional. Note that \mathcal{L}_0 is \mathbb{Z} -graded, as

$$(L_0)_{\bar{0}} = (\bar{\mathfrak{s}}_{(0)})_{\bar{0}} + (L_0) \cap \mathfrak{s}_{-2}, \quad (L_0)_{\bar{1}} = (\bar{\mathfrak{s}}_{(0)})_{\bar{1}} + (L_0) \cap \mathfrak{s}_{-1},$$

since the gradation of $\bar{\mathfrak{s}}/\bar{\mathfrak{s}}_{(0)} \simeq \mathfrak{m}$ is consistent and of depth 2. If $\mathcal{L}_0 \cap \mathfrak{s}_{-2} \neq 0$ then $\mathfrak{s}_{-2} \subset \mathcal{L}_0$. However a linearly compact Lie superalgebra $\bar{\mathfrak{s}}$ with a fundamental subalgebra \mathcal{L}_0 containing all even elements is necessarily finite-dimensional, since $\bar{\mathfrak{s}}$ is isomorphic to a subalgebra of $\text{der } \bigwedge(\bar{\mathfrak{s}}/\mathcal{L}_0)^*$ by the superalgebra version [34] of the Realization Theorem of Guillemin and Sternberg. We conclude that $\bar{\mathfrak{s}}_{(0)} \subset \mathcal{L}_0 \subsetneq \bar{\mathfrak{s}}_{(-1)}$.

We prove now that \mathcal{L}_0 is even primitive. All the even elements of the fundamental subalgebra \mathcal{L}_0 are exponentiable. If \mathcal{L}_0 is not even primitive, there exists a non-zero exponentiable element in $\mathfrak{s}_{-2} = V$. The subspace of exponentiable elements in V is invariant under all 0-degree automorphisms of \mathfrak{g} , in particular under the restrictions to V of the automorphisms $\{\psi_v\}_{v \in V}$. These generate $\text{O}(V)$, thus all elements of V , and of $\bar{\mathfrak{s}}_{\bar{0}}$, are exponentiable. Since there always exists a fundamental subalgebra containing all exponentiable elements [22], the Realization Theorem contradicts again the infinite-dimensionality of $\bar{\mathfrak{s}}$. \square

Proof of Theorem 3.4. Assume, by contradiction, that \mathfrak{g} is infinite dimensional. Let \mathfrak{s} be the minimal ideal of \mathfrak{g} described in Theorem 3.1. Note that \mathfrak{s} is an infinite-dimensional simple transitive prolongation of \mathfrak{m} and denote by $\bar{\mathfrak{s}}$ its completion.

Consider an even primitive subalgebra \mathcal{L}_0 with $\bar{\mathfrak{s}}_{(0)} \subset \mathcal{L}_0 \subsetneq \bar{\mathfrak{s}}_{(-1)}$, as established in Proposition 3.5, and a minimal \mathcal{L}_0 -invariant subspace \mathcal{L}_{-1} strictly containing \mathcal{L}_0 . By setting

$$\begin{aligned} \mathcal{L}_p &= \mathcal{L}_{p+1} + [\mathcal{L}_{p+1}, \mathcal{L}_{p+1}], & \text{for } p \leq -2, \\ \mathcal{L}_p &= \{X \in \mathcal{L}_{p-1} \mid [X, \mathcal{L}_{-1}] \subset \mathcal{L}_{p-1}\}, & \text{for } p \geq 1 \end{aligned}$$

we obtain a filtration of $\bar{\mathfrak{s}}$

$$\bar{\mathfrak{s}} = \mathcal{L}_{-\mu} \supset \cdots \supset \mathcal{L}_{-1} \supset \mathcal{L}_0 \supset \mathcal{L}_1 \cdots,$$

usually referred to as the Weisfeiler filtration [37]. We distinguish two cases.

Case $\mathcal{L}_0 = \bar{\mathfrak{s}}_{(0)}$. Let $S \subset \mathfrak{s}_{-1}$ be a non-zero irreducible \mathfrak{s}_0 -submodule of \mathfrak{s}_{-1} , and $\mathcal{L}_{-1} = \mathcal{L}_0 + S$. Since \mathcal{L}_0 is a maximal subalgebra, the subspace \mathcal{L}_{-1} generates $\bar{\mathfrak{s}}$.

We have $[\mathcal{L}_{-1}, \mathcal{L}_{-1}] \subset \mathcal{L}_{-1} + [S, S]$ and, since \mathcal{L}_{-1} is not a subalgebra, we obtain $[S, S] = V$. On the other hand

$$[\mathcal{L}_{-1}, [\mathcal{L}_{-1}, \mathcal{L}_{-1}]] \subset [\mathcal{L}_{-1}, \mathcal{L}_{-1}] + [\mathcal{L}_{-1}, V] \subset \bar{\mathfrak{s}}_{(0)} + S + V + [\mathfrak{s}_1, [S, S]]$$

and $[\mathfrak{s}_1, [S, S]] \subset [[\mathfrak{s}_1, S], S] \subset S$ imply that $\mathcal{L}_{-1} + [\mathcal{L}_{-1}, \mathcal{L}_{-1}]$ is a subalgebra. It follows that $\mathcal{L}_{-1} + [\mathcal{L}_{-1}, \mathcal{L}_{-1}] = \bar{\mathfrak{s}}$, W is \mathfrak{s}_0 -irreducible and equal to S , and $\mathcal{L}_{-1} = \bar{\mathfrak{s}}_{(-1)}$. The Weisfeiler filtration is given by $\mathcal{L}_p = \bar{\mathfrak{s}}_{(p)}$ for $p \geq -2$.

Case $\bar{\mathfrak{s}}_{(0)} \subsetneq \mathcal{L}_0$. Let \mathcal{L}_{-1} be a minimal \mathcal{L}_0 -invariant subspace strictly containing \mathcal{L}_0 , and $S = \mathcal{L}_{-1} \cap \mathfrak{s}_{-1}$. Note that $\mathcal{L}_0 \cap \mathfrak{s}_{-1}$ is an abelian subspace of \mathfrak{s}_{-1} . With an argument similar to the previous case, we get $S = \mathfrak{s}_{-1}$ and $\mathcal{L}_{-1} = \bar{\mathfrak{s}}$. Hence, the first few terms of the Weisfeiler filtration are $\mathcal{L}_{-1} = \bar{\mathfrak{s}}$, \mathcal{L}_0 , and

$$\mathcal{L}_1 = \bar{\mathfrak{s}}_{(2)} + \{D \in \mathfrak{s}_1 \mid [D, \mathfrak{s}_{-2}] \subset \mathcal{L}_0 \cap \mathfrak{s}_{-1}\} + \{D \in \mathfrak{h}_0 \mid [D, \mathfrak{s}_{-1}] \subset \mathcal{L}_0\}.$$

Let now \mathfrak{L} be the \mathbb{Z} -graded Lie superalgebra associated to the filtration \mathcal{L}_p . In the terminology of [22], \mathfrak{L} is a \mathbb{Z} -graded even transitive irreducible infinite-dimensional Lie superalgebra. Moreover, again by [22], the \mathfrak{L}_0 -module \mathfrak{L}_{-1} is *strongly transitive*, i.e. it is a faithful irreducible \mathfrak{L}_0 -module such that $[\mathfrak{L}_0, X] = \mathfrak{L}_{-1}$ for all non-zero even $X \in \mathfrak{L}_{-1}$. To conclude the proof of our theorem, we use the classification [22] of such Lie superalgebras and modules. We distinguish again the two cases above.

Case $\mathcal{L}_0 = \bar{\mathfrak{s}}_{(0)}$. The \mathbb{Z} -graded Lie superalgebra \mathfrak{L} coincides with \mathfrak{s} . Thus the \mathbb{Z} -gradation is consistent and the nonpositive part of \mathfrak{L} satisfies

- $\mathfrak{L}_{-3} = 0$;
- $\dim \mathfrak{L}_{-2} = \dim V \geq 3$;
- \mathfrak{L}_0 contains an ideal isomorphic to $\mathfrak{so}(V)$ acting on \mathfrak{L}_{-2} as the standard representation.

The even transitive irreducible consistent \mathbb{Z} -graded Lie superalgebras of depth $\mu \geq 2$ are listed in [22, Thm. 5.3] and described in detail in [9]. A case by case verification shows that such an \mathfrak{L} does not exist.

Case $\bar{\mathfrak{s}}_{(0)} \subsetneq \mathcal{L}_0$. The \mathbb{Z} -gradation of \mathfrak{L} is not consistent and the nonpositive part of \mathfrak{L} satisfies

- $\mathfrak{L}_{-2} = 0$;
- $(\mathfrak{L}_{-1})_{\bar{0}} = V$ has dimension greater or equal than 3;
- $(\mathfrak{L}_0)_{\bar{0}}$ contains an ideal isomorphic to $\mathfrak{so}(V)$ acting on $(\mathfrak{L}_{-1})_{\bar{0}}$ as the standard representation.

The strongly transitive modules with a non-zero even component are listed in [22, Thm 3.1] (and corrected in [8]). A case by case verification shows again that such an \mathfrak{L} does not exist. \square

4. CLASSIFICATION OF SIMPLE AND MAXIMAL PROLONGATIONS.

In section 3, we proved that the maximal transitive prolongation $\mathfrak{g} = \sum_{p \geq -2} \mathfrak{g}_p$ of a supertranslation algebra $\mathfrak{m} = \mathfrak{m}_{-2} + \mathfrak{m}_{-1} = V + W$ with $\dim V \geq 3$ is a finite-dimensional Lie superalgebra (Theorem 3.4) which is semisimple if $\mathfrak{g}_1 \neq 0$ (Theorem 3.1). In the latter case, \mathfrak{g} contains a unique minimal ideal \mathfrak{s} which is a simple prolongation of \mathfrak{m} (Theorem 3.1).

This section contains our main classification results: we first classify all possible simple prolongations (Theorem 4.5) and then derive the maximal transitive prolongations containing each of them (Theorem 4.11).

Remark 4.1. In the Lie algebra case, if \mathfrak{m} is a fundamental \mathbb{Z} -graded Lie algebra whose maximal transitive prolongation \mathfrak{g} is finite-dimensional, any prolongation \mathfrak{s} of \mathfrak{m} that is simple necessarily coincides with \mathfrak{g} [26]. The corresponding statement is not true in the Lie superalgebra case and one has to separately consider simple and maximal prolongations.

Finite-dimensional simple Lie superalgebras $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ are classified in [19] and split into two main families: *classical* Lie superalgebras, for which the adjoint action of \mathfrak{g}_0 on \mathfrak{g}_1 is completely reducible, and *Cartan* Lie superalgebras $W(m)$, $S(m)$, $\tilde{S}(m)$, $H(m)$ (finite-dimensional Lie superalgebras analogue to simple Lie algebras of vector fields). Classical Lie superalgebras in turn split into *basic* Lie superalgebras $A(m, n)$, $B(m, n)$, $C(n)$, $D(m, n)$, $D(2, 1; \alpha)$, $F(4)$, $G(3)$, for which there exists a non-degenerate even invariant supersymmetric bilinear form, and two *strange* families $P(m)$ and $Q(m)$.

Remark 4.2. The simple Lie superalgebras $S(3)$, $\tilde{S}(2)$ and $H(4)$ are isomorphic to the classical Lie superalgebras $P(2)$, $B(0, 1)$ and $A(1, 1)$ respectively. In our conventions, they are not Cartan Lie superalgebras.

We first prove that the Cartan and strange Lie superalgebras do not appear in our classification and then deal with basic Lie superalgebras.

4.1. Cartan type and strange Lie superalgebras. We briefly recall the description of Cartan Lie superalgebras and their possible \mathbb{Z} -gradations [20].

Let $W(m) = \{P(\xi_1, \dots, \xi_m)\partial_i \mid P \in \Lambda(m), 1 \leq i \leq m\}$ be the algebra of derivations of the Grassmann algebra $\Lambda(m)$ in m -generators ξ_1, \dots, ξ_m . The \mathbb{Z} -gradation of $W(m)$ of type (k_1, \dots, k_m) is defined by assigning degrees

$$\deg(\xi_i) = k_i, \quad \deg(\partial_i) = -k_i,$$

for integers k_1, \dots, k_m . Up to isomorphism, every gradation of $W(m)$ is of this form. Note that a gradation of type (k_1, \dots, k_m) is consistent precisely if all k_i 's are odd. The gradation of type $(1, \dots, 1)$ is usually called the *principal gradation* and, in this case, the subalgebra $W(m)_0 = \langle \xi_i \partial_j \mid 1 \leq i, j \leq m \rangle$ of 0-degree elements is identifiable with $\mathfrak{gl}(m, \mathbb{C})$. The Lie subalgebra

$$\mathfrak{sl}(m, \mathbb{C}) = \langle \xi_i \partial_j, \xi_i \partial_i - \xi_j \partial_j \mid 1 \leq i \neq j \leq m \rangle \quad (4.1)$$

is a Levi factor of the even part $W(m)_0$; it is \mathbb{Z} -graded in every gradation of type (k_1, \dots, k_m) .

We denote by $S(m)$ the subalgebra of divergence free derivations of $\Lambda(m)$ and by $\tilde{S}(m) = (1 + \xi_1 \cdots \xi_m)S(m)$ for m even the unique non-trivial simple

deformation of $S(m)$ (see [19] for more details). Finally, the Hamiltonian Lie superalgebra $H(m)$ is the derived ideal of the superalgebra preserving the symplectic form $\omega_m = \sum_{i=1}^m d\xi_i d\xi_{m+1-i}$, i.e.

$$H(m) = \left\langle \sum_{j=1}^m \partial_j f \partial_{m+1-j} + \partial_{m+1-j} f \partial_j \mid f \in \Lambda^k(n), 1 \leq k \leq m-1 \right\rangle.$$

Up to isomorphism, all \mathbb{Z} -gradations of the simple Lie superalgebras $S(m)$, $\tilde{S}(m)$ and $H(m)$ are induced by \mathbb{Z} -gradations of $W(m)$. More precisely, they are all obtained as follows (see [21], where there is a misprint in the $H(m)$ -case):

- i) every gradation of type (k_1, \dots, k_m) induces a gradation of $S(m)$,
- ii) every gradation of type (k_1, \dots, k_m) with $\sum_i k_i = 0$ induces a gradation of $\tilde{S}(m)$,
- iii) every gradation of type (k_1, \dots, k_m) with $k_i + k_{m+1-i} = k_j + k_{m+1-j}$ for all $1 \leq i, j \leq m$ induces a gradation of $H(m)$.

A gradation of $S(m)$, $\tilde{S}(m)$ or $H(m)$ is consistent if and only if it is induced by a consistent gradation of $W(m)$.

The subalgebra (4.1) is also a Levi factor of $S(m)_{\bar{0}}$ and $\tilde{S}(m)_{\bar{0}}$. The intersection $\mathfrak{sl}(m, \mathbb{C}) \cap H(m)$ is the Levi factor

$$\mathfrak{so}(m, \mathbb{C}) = \langle \xi_i \partial_{m+1-j} - \xi_j \partial_{m+1-i} \mid 1 \leq i < j \leq m \rangle \quad (4.2)$$

of $H(m)_{\bar{0}}$ and it is \mathbb{Z} -graded in every gradation of $H(m)$ of type (k_1, \dots, k_m) .

Proposition 4.3. *Let $\mathfrak{m} = V + W$ be a supertranslation algebra with $\dim V \geq 3$ and $\mathfrak{s} = \sum_{p \geq -2} \mathfrak{s}_p$ a simple prolongation of \mathfrak{m} . If \mathfrak{s}_0 contains an ideal isomorphic to $\mathfrak{so}(V)$, acting via the standard representation on \mathfrak{m}_{-2} and a multiple of the spinor representation on \mathfrak{m}_{-1} , then \mathfrak{s} is not a Lie superalgebra of Cartan type.*

Proof. Let $\mathfrak{s} = W(m)$ ($m \geq 3$), $S(m)$ ($m \geq 4$), $\tilde{S}(m)$ ($m \geq 4$ even) or $H(m)$ ($m \geq 5$). The gradation of \mathfrak{s} is induced by a gradation of $W(m)$ of type (k_1, \dots, k_m) and there exists a graded Levi decomposition $\mathfrak{s}_{\bar{0}} = \mathfrak{l} + \mathfrak{r}$ of the even part of \mathfrak{s} , where the simple Levi factor \mathfrak{l} is the one in (4.1) or (4.2). Since the gradation of \mathfrak{s} has depth 2, there are two possibilities for the gradation of \mathfrak{l} : either $\mathfrak{l} = \mathfrak{l}_0$ or $\mathfrak{l} = \mathfrak{l}_{-2} + \mathfrak{l}_0 + \mathfrak{l}_2$ with $\mathfrak{l}_{-2} \neq 0$. We treat the two cases separately.

Case $\mathfrak{l} = \mathfrak{l}_0$. The subalgebra \mathfrak{l} is also a Levi factor of \mathfrak{s}_0 . Being simple, it coincides with $\mathfrak{so}(V)$. This implies that $\mathfrak{s} = W(4)$, $S(4)$, $\tilde{S}(4)$ and $\dim V = 6$, or $\mathfrak{s} = H(m)$ and $\dim V = m \geq 5$. The condition $\mathfrak{l} \subset \mathfrak{s}_0$ also implies that the m -tuple (k_1, \dots, k_m) is an integer multiple of $(1, \dots, 1)$. The gradation of \mathfrak{s} is fundamental, forcing $(k_1, \dots, k_m) = \pm(1, \dots, 1)$, and of depth 2, leaving only the possibility $(k_1, \dots, k_m) = (-1, \dots, -1)$.

This gradation has depth 3 for $W(4)$. For $S(4)$ it has depth 2 but $\dim \mathfrak{s}_{-2} = 10 \neq 6$. For $\tilde{S}(4)$ the gradation is not admissible. For $H(m)$ it has depth $m-3$ and, in the special case $H(5)$, the $\mathfrak{so}(V)$ module \mathfrak{s}_{-1} is isomorphic to $\Lambda^3 V$, which is not a multiple of the spinor module.

Case $\mathfrak{l} = \mathfrak{l}_{-2} + \mathfrak{l}_0 + \mathfrak{l}_2$ with $\mathfrak{l}_{-2} \neq 0$. If $\mathfrak{l}_{-2} \subsetneq V$, then $\mathfrak{r}_{-2} \neq 0$ and this would imply $\mathfrak{r}_{-2} = V$, a contradiction. Hence \mathfrak{l}_0 acts irreducibly and conformally on $\mathfrak{l}_{-2} = V$. Since $\mathfrak{l}_2 \simeq V^*$, one gets that $\mathfrak{l}_0 \simeq \mathfrak{co}(V)$ and \mathfrak{l} is isomorphic to $\mathfrak{so}(d+2, \mathbb{C})$, where $d = \dim V$.

This happens precisely when $\mathfrak{l} = \mathfrak{sl}(4, \mathbb{C}) \simeq \mathfrak{so}(6, \mathbb{C})$ and $\mathfrak{s} = W(4), S(4), \tilde{S}(4)$ or when $\mathfrak{l} = \mathfrak{so}(m, \mathbb{C})$ and $\mathfrak{s} = H(m)$.

In the former case, the conditions that \mathfrak{s} has depth 2 and $\dim V = 4$ rule out all admissible gradations of $\mathfrak{s} = W(4), S(4), \tilde{S}(4)$.

In the latter case, it is straightforward to check that $H(m)$ with $m \geq 5$ does not admit any consistent gradation of depth 2 with $H(m)_2 \neq 0$. \square

Having dealt with Lie superalgebras of Cartan type, we turn now to the two families of strange Lie superalgebras.

Proposition 4.4. *Let $\mathfrak{m} = V + W$ be a supertranslation algebra with $\dim V \geq 3$ and $\mathfrak{s} = \sum_{p \geq -2} \mathfrak{s}_p$ a simple prolongation of \mathfrak{m} . If \mathfrak{s}_0 contains an ideal isomorphic to $\mathfrak{so}(V)$, acting via the standard representation on \mathfrak{m}_{-2} and a multiple of the spinor representation on \mathfrak{m}_{-1} , then \mathfrak{s} is not a strange Lie superalgebra $P(m)$ or $Q(m)$ ($m \geq 1$).*

Proof. Assume $\mathfrak{s} = P(m) = P(m)_{\bar{0}} + P(m)_{\bar{1}} \simeq \mathfrak{sl}(m+1) + \Pi(\vee^2 \mathbb{C}^m + \Lambda^2 \mathbb{C}^{m*})$. The even part is graded $\mathfrak{s}_{\bar{0}} = \mathfrak{s}_{-2} + \mathfrak{s}_0 + \mathfrak{s}_2$ and \mathfrak{s}_0 contains $\mathfrak{so}(V)$ as an ideal acting with the standard representations on $\mathfrak{s}_{-2} = V$. The only possibility is that $m = 3$ and $\mathfrak{sl}(4, \mathbb{C}) \simeq \mathfrak{so}(6, \mathbb{C})$ is the Cartan prolongation of $(\mathbb{C}^4, \mathfrak{co}(4, \mathbb{C}))$. However the classification in [21] implies that all consistent gradations of $P(3)$ with $P(3)_0 \simeq \mathfrak{co}(4, \mathbb{C})$ have depth at least 3.

Finally, $Q(m)$ does not admit any consistent gradation. \square

4.2. Basic Lie superalgebras. We briefly recall some notions about basic Lie superalgebras, their Dynkin diagrams and their \mathbb{Z} -gradations.

A simple Lie superalgebra $\mathfrak{s} = \mathfrak{s}_{\bar{0}} + \mathfrak{s}_{\bar{1}}$ is called *basic* if the even part $\mathfrak{s}_{\bar{0}}$ is a reductive Lie algebra and there exists an even non-degenerate invariant supersymmetric bilinear form $B : \mathfrak{s} \otimes \mathfrak{s} \rightarrow \mathbb{C}$. There are four families $A(m, n)$, $B(m, n)$, $C(n)$, $D(m, n)$, a family $D(2, 1; \alpha)$ of deformations of $D(2, 1)$ and the exceptional cases $F(4)$ and $G(3)$; the list can be found in [19]. Note that the form B is unique up to constant and it coincides with the Killing form of \mathfrak{s} , except for the cases $A(m, m)$, $D(m+1, m)$, $D(2, 1; \alpha)$.

For later use, we give in Table 1 the description of the even Lie subalgebra $\mathfrak{s}_{\bar{0}}$ of \mathfrak{s} and its representation on the odd part $\mathfrak{s}_{\bar{1}}$.

\mathfrak{s}	$\mathfrak{s}_{\bar{0}}$	$\mathfrak{s}_{\bar{1}}$
$A(m, n)$ $m, n \geq 1, m \neq n$	$\mathfrak{sl}(m+1, \mathbb{C}) \oplus \mathfrak{sl}(n+1, \mathbb{C}) \oplus \mathbb{C}Z$	$\mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1*} + \mathbb{C}^{m+1*} \otimes \mathbb{C}^{n+1}$
$A(n, n)$ $n \geq 1$	$\mathfrak{sl}(n+1, \mathbb{C}) \oplus \mathfrak{sl}(n+1, \mathbb{C})$	$\mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1*} + \mathbb{C}^{n+1*} \otimes \mathbb{C}^{n+1}$
$B(m, n)$ $n \geq 1$	$\mathfrak{so}(2m+1, \mathbb{C}) \oplus \mathfrak{sp}(2n, \mathbb{C})$	$\mathbb{C}^{2m+1} \otimes \mathbb{C}^{2n}$
$C(n)$ $n \geq 2$	$\mathfrak{so}(2, \mathbb{C}) \oplus \mathfrak{sp}(2n-2, \mathbb{C})$	$\mathbb{C}^2 \otimes \mathbb{C}^{2n-2}$
$D(m, n)$ $m \geq 2, n \geq 1$	$\mathfrak{so}(2m, \mathbb{C}) \oplus \mathfrak{sp}(2n, \mathbb{C})$	$\mathbb{C}^{2m} \otimes \mathbb{C}^{2n}$
$D(2, 1; \alpha)$ $\alpha \neq 0, -1$	$\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$	$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$
$F(4)$	$\mathfrak{so}(7, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$	$\mathbb{S} \otimes \mathbb{C}^2$
$G(3)$	$G_2 \oplus \mathfrak{sl}(2, \mathbb{C})$	$\mathbb{C}^7 \otimes \mathbb{C}^2$

TABLE 1.

Dynkin diagrams. Basic Lie superalgebras can be described by means of Cartan matrices, more precisely they are the quotients of indecomposable finite-dimensional *contragredient* Lie superalgebras by their center [19, 20]. In all cases, the center is trivial with the exception of $A(n, n)$; the contragredient Lie superalgebra $sl(n+1|n+1)$ has a one-dimensional center and $A(n, n)$ is the quotient $psl(n+1|n+1)$.

A convenient way to describe Cartan matrices is provided by Dynkin diagrams. They have been first introduced for Lie superalgebras in [19], however we will use the slightly different conventions given by [11, 12]. We recall here only the facts that we need and refer to those texts for more details.

Let \mathfrak{g} be an indecomposable finite-dimensional contragredient Lie superalgebra, \mathfrak{t} a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$ and $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$ the associated root system. Then $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}}$ decompose into the sum of root spaces and a root α is called *even* (resp. *odd*) if the root space $\mathfrak{g}_{\bar{0}}^{\alpha}$ (resp. $\mathfrak{g}_{\bar{0}}^{\alpha}$) is non-zero. Every root is either even or odd and the root spaces are one-dimensional, except in the case $\mathfrak{g} = sl(2|2)$ where all four odd roots have two-dimensional eigenspaces. Many properties of root systems of Lie algebras remain true for basic Lie superalgebras [20, Prop. 5.3]. In particular any decomposition $\Delta = \Delta^+ \cup -\Delta^+$ into positive and negative roots determines a system $\Pi = \{\alpha_1, \dots, \alpha_r\}$ of simple positive roots. Every positive root $\alpha \in \Delta^+$ can be written in a canonical way as a sum $\alpha = \sum_i b_i \alpha_i$, with positive integer coefficients b_i .

The Weyl group of $\mathfrak{g}_{\bar{0}}$ acts on the set of simple root systems. In contrast with the Lie algebra case this action is not transitive, and hence different simple root systems of the same basic Lie superalgebra may not be conjugated. To each orbit of the Weyl group one can associate a Dynkin diagram as follows. Each simple root α corresponds to a node which is colored *white*

if α is even, *gray* if α is odd and B -isotropic, or *black* if α is odd and non-isotropic. Consider a symmetric Cartan matrix (a_{ij}) associated to Π [24]; the i -th and j -th nodes of the Dynkin diagram are joined by η_{ij} lines where

- $\eta_{ij} = \frac{2|a_{ij}|}{\min(|a_{ii}|, |a_{jj}|)}$ if $a_{ii} \neq 0$ and $a_{jj} \neq 0$,
- $\eta_{ij} = \frac{2|a_{ij}|}{\min(|a_{ii}|, 2)}$ if $a_{ii} \neq 0$ and $a_{jj} = 0$,
- $\eta_{ij} = |a_{ij}|$ if $a_{ii} = a_{jj} = 0$.

The lines have an arrow pointing from α_i to α_j if $\eta_{ij} \geq 2$ and

- $a_{ii} \neq 0$, $a_{jj} \neq 0$ and $|a_{ii}| > |a_{jj}|$, or
- $a_{ii} = 0$, $a_{jj} \neq 0$ and $|a_{jj}| < 2$, or
- $a_{ii} \neq 0$, $a_{jj} = 0$ and $|a_{ii}| > 2$.

Finally, the i -th node is marked with the coefficient c_i of the expression of the highest root $\alpha_{\max} = \sum_i c_i \alpha_i$ as sum of simple roots. For $D(2, 1; \alpha)$ additional data is required, but our arguments do not use it and we omit it.

A list of all possible Dynkin diagrams is contained in [11, Table 2].

Fundamental consistent \mathbb{Z} -gradations. Let \mathfrak{s} be a finite-dimensional contragredient Lie superalgebra different from $sl(2|2)$, with a Cartan subalgebra \mathfrak{t} , a root system Δ and a simple root system Π . Let $\deg \alpha_i = 0$ if $\alpha_i \in \Pi$ is even and $\deg \alpha_i = 1$ if $\alpha_i \in \Pi$ is odd, and extend the definition to all roots by

$$\deg(\sum b_i \alpha_i) = \sum b_i \deg(\alpha_i).$$

The gradation of \mathfrak{g} given by

$$\begin{aligned} \mathfrak{g}_0 &= \mathfrak{t} + \sum_{\substack{\alpha \in \Delta \\ \deg \alpha = 0}} \mathfrak{g}^\alpha, \\ \mathfrak{g}_p &= \sum_{\substack{\alpha \in \Delta \\ \deg \alpha = p}} \mathfrak{g}^\alpha. \end{aligned}$$

is consistent and fundamental.

By a result of Kac [21] all possible consistent fundamental gradations of \mathfrak{s} are equivalent to one of this form, for some choice of Π . In particular every Dynkin diagram canonically describes a unique consistent and fundamental gradation. The depth of \mathfrak{s} is equal to the degree of the maximal root. The subalgebra \mathfrak{s}_0 is a reductive Lie algebra, the Dynkin diagram of its semisimple ideal is obtained from the Dynkin diagram of \mathfrak{s} by removing all gray and black nodes, and any line issuing from them.

Main classification result. We can now state and prove the following theorem.

Theorem 4.5. *Let \mathfrak{s} be a basic Lie superalgebra satisfying the following properties:*

- (1) $\mathfrak{s} = \sum_{p \geq -2} \mathfrak{s}_p$ is a consistent \mathbb{Z} -graded Lie superalgebra of depth 2 with $\dim \mathfrak{s}_{-2} \geq 3$,
- (2) the gradation is fundamental and transitive,

- (3) *there exists an identification between an orthogonal space V and \mathfrak{s}_{-2} and an ideal of \mathfrak{s}_0 whose action on $\mathfrak{s}_{-2} \simeq V$ is equivalent to the standard representation of $\mathfrak{so}(V)$,*
- (4) *the adjoint action of $\mathfrak{so}(V)$ on \mathfrak{s}_{-1} is equivalent to a sum of spinor or semi-spinor representations.*

Then \mathfrak{s} is isomorphic to one of the Lie superalgebras listed in Table 2, with the consistent \mathbb{Z} -gradation determined by the Dynkin diagram.

\mathfrak{s}	Dynkin diagram	\mathfrak{s}_{-2}	\mathfrak{s}_{-1}	\mathfrak{s}_0
$A(3, m)$ $m \neq 3$	$\overset{1}{\circ} - \overset{1}{\bullet} - \dots - \overset{1}{\bullet} - \overset{1}{\circ}$	\mathbb{C}^4	$\mathbb{S}^+ \otimes \mathbb{C}^{m+1} + \mathbb{S}^- \otimes \mathbb{C}^{m+1*}$	$\mathfrak{so}_4 \oplus \mathbb{C}E \oplus \mathfrak{gl}_{m+1}$
$A(3, 3)$	$\overset{1}{\circ} - \overset{1}{\bullet} - \dots - \overset{1}{\bullet} - \overset{1}{\circ}$	\mathbb{C}^4	$\mathbb{S}^+ \otimes \mathbb{C}^4 + \mathbb{S}^- \otimes \mathbb{C}^{4*}$	$\mathfrak{so}_4 \oplus \mathbb{C}E \oplus \mathfrak{sl}_4$
$B(0, 2)$	$\overset{2}{\circ} \rightleftarrows \overset{2}{\bullet}$	\mathbb{C}^3	\mathbb{S}	$\mathfrak{so}_3 \oplus \mathbb{C}E$
$B(m, 2)$ $m \geq 1$	$\overset{2}{\circ} - \overset{2}{\bullet} - \dots \rightleftarrows \overset{2}{\circ}$	\mathbb{C}^3	$\mathbb{S} \otimes \mathbb{C}^{2m+1}$	$\mathfrak{so}_3 \oplus \mathbb{C}E \oplus \mathfrak{so}_{2m+1}$
$C(3)$	$\begin{array}{c} \overset{1}{\bullet} \\ \diagup \quad \diagdown \\ \overset{2}{\circ} - \overset{2}{\bullet} \\ \diagdown \quad \diagup \\ \overset{1}{\bullet} \end{array}$	\mathbb{C}^3	$\mathbb{S} \otimes \mathbb{C}^2$	$\mathfrak{so}_3 \oplus \mathbb{C}E \oplus \mathfrak{so}_2$
$D(m, 2)$ $m \geq 2$	$\overset{2}{\circ} - \overset{2}{\bullet} - \dots \begin{array}{l} \nearrow \overset{1}{\circ} \\ \searrow \overset{1}{\circ} \end{array}$	\mathbb{C}^3	$\mathbb{S} \otimes \mathbb{C}^{2m}$	$\mathfrak{so}_3 \oplus \mathbb{C}E \oplus \mathfrak{so}_{2m}$
$D(4, m)$ $m \geq 1$	$\overset{1}{\circ} - \overset{2}{\circ} - \overset{2}{\circ} - \overset{2}{\bullet} - \dots \rightleftarrows \overset{1}{\circ}$	\mathbb{C}^6	$\mathbb{S}^+ \otimes \mathbb{C}^{2m}$	$\mathfrak{so}_6 \oplus \mathbb{C}E \oplus \mathfrak{sp}_{2m}$
$F(4)$	$\overset{1}{\circ} \rightleftarrows \overset{2}{\bullet} - \overset{3}{\circ} \rightleftarrows \overset{2}{\circ}$	\mathbb{C}^5	$\mathbb{S} \otimes \mathbb{C}^2$	$\mathfrak{so}_5 \oplus \mathbb{C}E \oplus \mathfrak{sl}_2$

TABLE 2.

Remark 4.6. We recall that the simple Lie superalgebra $A(n, n)$ ($n \geq 1$) is defined as the quotient of $sl(n+1|n+1)$ by its one-dimensional center. Moreover the derivations of $sl(n+1|n+1)$ and of $A(n, n)$ are all induced by the adjoint action of elements in $gl(n+1|n+1)$; in particular the center of $sl(n+1|n+1)$ has degree zero for any \mathbb{Z} -gradation of $sl(n+1|n+1)$ and \mathbb{Z} -gradations of $sl(n+1|n+1)$ are in natural correspondence with those of $A(n, n)$.

Hence the Dynkin diagram in the $A(3, 3)$ -row of Table 2 has to be understood as the Dynkin diagram of the contragredient Lie superalgebra $sl(4|4)$ and the consistent gradation of $A(3, 3)$ as the one induced from $sl(4|4)$.

Proof. First we look at the even part $\mathfrak{s}_{\bar{0}}$ of the \mathbb{Z} -graded Lie superalgebra \mathfrak{s} . It is a reductive Lie algebra $\mathfrak{s}_{\bar{0}} = [\mathfrak{s}_{\bar{0}}, \mathfrak{s}_{\bar{0}}] + \mathfrak{z}$ where $[\mathfrak{s}_{\bar{0}}, \mathfrak{s}_{\bar{0}}] = \oplus_i \mathfrak{s}^i$ is a sum of simple ideals and \mathfrak{z} is the center of $\mathfrak{s}_{\bar{0}}$. The \mathbb{Z} -gradation of \mathfrak{s} induces

\mathbb{Z} -gradations $\mathfrak{s}^i = \mathfrak{s}_{-2}^i + \mathfrak{s}_0^i + \mathfrak{s}_2^i$ on each simple factor and on the center $\mathfrak{z} = \sum_{p \geq -1} \mathfrak{z}_{2p}$.

By hypothesis (3), $\mathfrak{z}_{-2} = 0$. The invariant bilinear form B of \mathfrak{s} is nondegenerate on \mathfrak{s}_0 and hence on each \mathfrak{s}^i and on \mathfrak{z} . In all cases where \mathfrak{z} is not zero, B coincides with the Killing form of \mathfrak{s} . It follows that the \mathfrak{z}_p and \mathfrak{z}_{-p} are dual to each other, and then $\mathfrak{z} = \mathfrak{z}_0$.

Let $d = \dim V$. By hypothesis (1) and (3), we can assume without loss of generality that $\mathfrak{s}_{-2} \subset \mathfrak{s}^1$ and $\mathfrak{s}^i \subset \mathfrak{s}_0$ for all $i \geq 2$. In particular the ideal $\mathfrak{so}(V) \simeq \mathfrak{so}(d, \mathbb{C})$ of \mathfrak{s}_0 is contained in \mathfrak{s}^1 . The ideal \mathfrak{s}^1 of \mathfrak{s}_0 has then a gradation $\mathfrak{s}^1 = \mathfrak{s}_{-2}^1 + \mathfrak{s}_0^1 + \mathfrak{s}_2^1 = V + (\mathfrak{so}(V) + \mathfrak{e}) + V^*$ where \mathfrak{e} is an ideal of \mathfrak{s}_0^1 . It follows that $\mathfrak{s}_0^1 \simeq \mathfrak{co}(V)$ and $\mathfrak{s}^1 \simeq \mathfrak{so}(d+2, \mathbb{C})$.

From the description of the even part of the basic Lie superalgebras, \mathfrak{s} must be one of the following: $B(m, 2)$, $C(3)$ or $D(m, 2)$ for $d = 3$; $A(3, m)$ for $d = 4$; $F(4)$ for $d = 5$; $B(m, n)$ for $d = 2m - 1$; $D(m, n)$ for $d = 2m - 2$.

By hypothesis (4), the representation of $\mathfrak{so}(V)$ on the odd part \mathfrak{s}_1 contains at least one factor of half-spin type. This implies that also the representation of $\mathfrak{s}^1 \simeq \mathfrak{so}(d+2, \mathbb{C})$ on \mathfrak{s}_1 contains a factor of half-spin type (in the case $d = 6$, since the semi-spinor and the standard representations of $\mathfrak{so}(8, \mathbb{C})$ are related by triality, this condition must hold true for *some* identification $\mathfrak{s}^1 \simeq \mathfrak{so}(8, \mathbb{C})$).

By looking at Table 1, one obtains exactly the simple Lie superalgebras \mathfrak{s} listed in the first column of Table 2.

To conclude the proof, we first determine all \mathbb{Z} -gradations of depth 2 of the \mathfrak{s} in the above list which satisfy hypotheses (1)–(3) and such that $[\mathfrak{s}_0, \mathfrak{s}_0] = \mathfrak{s}_0^1 \oplus \bigoplus_{i \geq 2} \mathfrak{s}^i$. A case by case analysis of [11, Table 2] reveals that the Dynkin diagrams satisfying the previous conditions are exactly those displayed in Table 2.

Finally, by using the explicit description in [12] of the root systems associated to the Dynkin diagrams of Table 2, it is a tedious but straightforward task to check that all the listed gradations also satisfy hypothesis (4). \square

We now explicitly describe the negatively graded parts $\mathfrak{s}_{-2} + \mathfrak{s}_{-1}$ of the \mathbb{Z} -graded Lie superalgebras listed in Table 2.

In all cases except $D(4, m)$, the negatively graded part is isomorphic to a supertranslation algebra $\mathfrak{m}(V, W, \beta)$: we now exhibit a nondegenerate bilinear form β on $\mathfrak{m}_{-1} = \mathbb{S} + \cdots + \mathbb{S}$ satisfying (B1), (B2), (B3) and such $\mathfrak{m} = \mathfrak{m}_{-2} + \mathfrak{m}_{-1}$ with the bracket (2.1) is isomorphic to $\mathfrak{s}_{-2} + \mathfrak{s}_{-1}$.

Example 4.7. $A(3, m)$. As an $(\mathfrak{so}(4, \mathbb{C}) \oplus \mathfrak{sl}(m+1, \mathbb{C}))$ -module, \mathfrak{s}_{-1} is isomorphic to $(\mathbb{S}^+ \otimes \mathbb{C}^{m+1}) + (\mathbb{S}^- \otimes \mathbb{C}^{m+1*})$. The bracket is induced by a $(\mathfrak{so}(4, \mathbb{C}) \oplus \mathfrak{sl}(m+1, \mathbb{C}))$ -equivariant map $(\mathbb{S}^+ \otimes \mathbb{C}^{m+1}) \times (\mathbb{S}^- \otimes \mathbb{C}^{m+1*}) \rightarrow V$. Hence, there exists a $\mathfrak{so}(4, \mathbb{C})$ -equivariant map $\Gamma : \mathbb{S}^+ \times \mathbb{S}^- \rightarrow V$ such that

$$[s^+ \otimes c, s^- \otimes c^*] = \Gamma(s^+, s^-)c^*(c),$$

for any $s^\pm \in \mathbb{S}^\pm$, $c \in \mathbb{C}^{m+1}$ and $c^* \in \mathbb{C}^{m+1*}$. By the results of [1], the map Γ is uniquely determined by a nondegenerate bilinear form $b : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{C}$ with invariants $(\tau, \sigma) = (-, -)$ by $(\Gamma(s^+, s^-), v) = b(v \cdot s^+, s^-)$.

Identifying \mathbb{C}^{m+1} and \mathbb{C}^{m+1*} via a nondegenerate symmetric bilinear form

δ on \mathbb{C}^{m+1} , we define a Clifford multiplication $\mathfrak{s}_{-2} \otimes \mathfrak{s}_{-1} \rightarrow \mathfrak{s}_{-1}$ by:

$$v \cdot (s \otimes c) = (v \cdot s) \otimes c, \quad (4.3)$$

and a nondegenerate $\mathfrak{so}(4, \mathbb{C})$ -invariant bilinear form $\beta : \mathfrak{s}_{-1} \times \mathfrak{s}_{-1} \rightarrow \mathbb{C}$:

$$\beta(s \otimes c, t \otimes d) = b(s, t)\delta(c, d).$$

Direct computations show that \cdot and β satisfy (B1), (B2), (B3).

Example 4.8. $B(m, 2)$ ($m \geq 0$), $C(3)$ and $D(m, 2)$ ($m \geq 2$). This is the orthosymplectic case $\mathfrak{s} = osp(n|4)$ with $n \geq 1$. In this case, $\mathfrak{s}_{-1} \simeq \mathbb{S} \otimes \mathbb{C}^n$ as an $(\mathfrak{so}(3, \mathbb{C}) \oplus \mathfrak{so}(n, \mathbb{C}))$ -module and the bracket is given by an $(\mathfrak{so}(3, \mathbb{C}) \oplus \mathfrak{so}(n, \mathbb{C}))$ -equivariant map $\vee^2 \mathbb{S} \otimes \vee^2 \mathbb{C}^n \rightarrow V$. There exists a $\mathfrak{so}(3, \mathbb{C})$ -equivariant map $\Gamma : \mathbb{S} \vee \mathbb{S} \rightarrow V$ and a non-degenerate symmetric $\mathfrak{so}(n, \mathbb{C})$ -invariant bilinear form δ on \mathbb{C}^n , given by the matrix with anti-diagonal entries equal to 1, such that

$$[s \otimes c, t \otimes d] = \Gamma(s, t)\delta(c, d),$$

for any $s, t \in \mathbb{S}$ and $c, d \in \mathbb{C}^n$. The map Γ is uniquely determined by a nondegenerate bilinear form $b : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{C}$ with invariants $(\tau, \sigma) = (-, -)$, via $(\Gamma(s, t), v) = b(v \cdot s, t)$.

We again define a Clifford multiplication by (4.3) and a nondegenerate $(\mathfrak{so}(3, \mathbb{C}) \oplus \mathfrak{so}(n, \mathbb{C}))$ -invariant bilinear form $\beta = b \otimes \delta : \mathfrak{s}_{-1} \times \mathfrak{s}_{-1} \rightarrow \mathbb{C}$.

Example 4.9. $F(4)$. In this case, $\mathfrak{s}_{-1} \simeq \mathbb{S} \otimes \mathbb{C}^2$ as an $(\mathfrak{so}(5, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}))$ -module and the bracket is given by a $(\mathfrak{so}(5, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}))$ -equivariant map $\Lambda^2 \mathbb{S} \otimes \Lambda^2 \mathbb{C}^2 \rightarrow V$. Hence, there exists a $\mathfrak{so}(5, \mathbb{C})$ -equivariant map $\Gamma : \mathbb{S} \wedge \mathbb{S} \rightarrow V$ and a nondegenerate $\mathfrak{sl}(2, \mathbb{C})$ -invariant bilinear form ω on \mathbb{C}^2 such that

$$[s \otimes c, t \otimes d] = \Gamma(s, t)\omega(c, d),$$

for any $s, t \in \mathbb{S}$ and $c, d \in \mathbb{C}^2$. The map Γ is uniquely determined by a nondegenerate bilinear form $b : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{C}$ with invariants $(\tau, \sigma) = (+, -)$, via $([s, t], v) = b(v \cdot s, t)$. We define a Clifford multiplication by (4.3) and a nondegenerate $(\mathfrak{so}(5, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}))$ -invariant bilinear form $\beta = b \otimes \omega : \mathfrak{s}_{-1} \times \mathfrak{s}_{-1} \rightarrow \mathbb{C}$.

In the $D(4, m)$ case the negatively graded part of \mathfrak{s} is not a supertranslation algebra, but it nevertheless admits a similar description in terms of semi-spinor modules.

Example 4.10. $D(4, m)$ ($m \geq 1$). In this case, $\mathfrak{s}_{-1} \simeq \mathbb{S}^+ \otimes \mathbb{C}^{2m}$ as a $(\mathfrak{so}(6, \mathbb{C}) \oplus \mathfrak{sp}(2m, \mathbb{C}))$ -module and the bracket is given by a $(\mathfrak{so}(6, \mathbb{C}) \oplus \mathfrak{sp}(2m, \mathbb{C}))$ -equivariant map $\Lambda^2 \mathbb{S}^+ \otimes \Lambda^2 \mathbb{C}^{2m} \rightarrow V$. There exists a $\mathfrak{so}(6, \mathbb{C})$ -equivariant map $\Gamma : \mathbb{S}^+ \wedge \mathbb{S}^+ \rightarrow V$ and a nondegenerate $\mathfrak{sp}(2m, \mathbb{C})$ -invariant bilinear form ω on \mathbb{C}^{2m} such that

$$[s^+ \otimes c, t^+ \otimes d] = \Gamma(s^+, t^+)\omega(c, d),$$

for any $s^+, t^+ \in \mathbb{S}^+$ and $c, d \in \mathbb{C}^{2m}$.

4.3. The maximal transitive prolongation. So far we have proved (Theorems 3.1, 3.4, 4.5) that the maximal transitive prolongation \mathfrak{g} of a supertranslation algebra $\mathfrak{m} = \mathfrak{m}_{-2} + \mathfrak{m}_{-1} = V + W$ with $\dim V \geq 3$ either satisfies $\mathfrak{g}_p = 0$ for all $p \geq 1$ or

- \mathfrak{g} is a finite-dimensional semisimple Lie superalgebra,
- \mathfrak{g} has a unique minimal ideal \mathfrak{s} which is a simple prolongation of \mathfrak{m} ,
- \mathfrak{s} is one of the \mathbb{Z} -graded Lie superalgebras in Table 2.

In this section, for each choice of \mathfrak{s} in Table 2 we determine the corresponding maximal prolongation \mathfrak{g} . It turns out that $\mathfrak{s} = \mathfrak{g}$ except in the case $\mathfrak{s} = A(3, 3)$, $\mathfrak{g} = \text{der}A(3, 3)$.

Theorem 4.11. *Let $\mathfrak{m} = V + W$ be a supertranslation algebra with $\dim V \geq 3$, and \mathfrak{g} the maximal transitive prolongation of \mathfrak{m} . If $\mathfrak{g}_1 \neq 0$, then \mathfrak{g} is one of the Lie superalgebras listed in Table 3.*

\mathfrak{g}	Dynkin diagram	$\dim V$	$\dim W$	N	\mathfrak{h}_0
$A(3, m)$ $m \neq 3$	$\begin{array}{c} 1 \quad 1 \quad \dots \quad 1 \quad 1 \\ \circ - \bullet - \dots - \bullet - \circ \end{array}$	4	$4N$	$m + 1$	\mathfrak{gl}_{m+1}
$\text{der}(A(3, 3))$	$\begin{array}{c} 1 \quad 1 \quad \dots \quad 1 \quad 1 \\ \circ - \bullet - \dots - \bullet - \circ \end{array}$	4	$4N$	4	\mathfrak{gl}_4
$B(0, 2)$	$\begin{array}{c} 2 \quad 2 \\ \circ \rightleftarrows \bullet \end{array}$	3	2	1	0
$B(m, 2)$ $m \geq 1$	$\begin{array}{c} 2 \quad 2 \quad \dots \quad 2 \\ \circ - \bullet - \dots - \bullet \end{array}$	3	$2N$	$2m + 1$	\mathfrak{so}_{2m+1}
$C(3)$	$\begin{array}{c} \quad \quad 1 \\ \quad \quad \bullet \\ \quad \diagup \quad \diagdown \\ 2 \quad \quad 1 \\ \circ \quad \quad \bullet \end{array}$	3	$2N$	2	\mathfrak{so}_2
$D(m, 2)$ $m \geq 2$	$\begin{array}{c} 2 \quad 2 \quad \dots \quad \quad \quad 1 \\ \circ - \bullet - \dots - \quad \quad \quad \diagup \quad \diagdown \\ \quad \quad \quad \quad \quad \quad \quad 1 \\ \quad \quad \quad \quad \quad \quad \quad \circ \end{array}$	3	$2N$	$2m$	\mathfrak{so}_{2m}
$F(4)$	$\begin{array}{c} 1 \quad 2 \quad 3 \quad 2 \\ \circ \rightleftarrows \bullet - \circ \rightleftarrows \circ \end{array}$	5	$4N$	2	\mathfrak{sl}_2

TABLE 3.

Proof. By Theorem 3.1, \mathfrak{g} is semisimple and contains a unique minimal ideal \mathfrak{s} , which is a simple prolongation of \mathfrak{m} with $\mathfrak{so}(V) \subset \mathfrak{s}_0 \subset \mathfrak{g}_0$ and $\mathfrak{g}_1 = \mathfrak{s}_1$. By Theorem 3.4, \mathfrak{g} and \mathfrak{s} are finite-dimensional. It follows that \mathfrak{s} is one of the Lie superalgebras in Table 2, different from $D(4, m)$.

We recall that the socle of \mathfrak{g} is the sum of all nonzero minimal ideals of \mathfrak{g} , and it is proved in [19, 7] that it is of the form $\sum_{i=1}^r \mathfrak{s}^i \otimes \Lambda(m_i)$, where \mathfrak{s}^i is a simple Lie superalgebra for all $i = 1, \dots, r$. In our case, the socle of \mathfrak{g} equals \mathfrak{s} , and then $r = 1$ and $m_1 = 0$.

From the characterization of semisimple Lie superalgebras in [19, 7] it follows that $\mathfrak{s} \subset \mathfrak{g} \subset \text{der } \mathfrak{s}$.

To conclude, observe that $\mathfrak{s} = \text{der } \mathfrak{s}$ for all Lie superalgebras in Table 2, with the exception of $\text{der } A(3, 3) \simeq pgl(4|4)$ [19]. It is straightforward to check that the negatively graded part of $\text{der } A(3, 3)$ coincides with \mathfrak{m} and that $\text{der } A(3, 3)$ is a transitive prolongation of \mathfrak{m} . \square

5. THE CLASSIFICATION

In this section we explicitly describe all the maximal transitive prolongations of supertranslation algebras for all possible dimensions of V and all $N \geq 1$. We include the cases $\dim V = 1$ or 2 : Theorem 3.4 does not apply, and the maximal transitive prolongation turns out to be infinite-dimensional.

In the next Theorem, we denote by $K(m, n)$ the infinite-dimensional contact superalgebra in super-dimension $(m|n)$, with $m = 2k + 1$ (see e.g. [22] for more details). By [9, Prop. 3.1.3], $K(m, n)$ with its principal \mathbb{Z} -gradation is the maximal transitive prolongation of its negatively graded part $K(m, n)_{-2} + K(m, n)_{-1} \simeq \mathbb{C}^{1|0} + \mathbb{C}^{2k|n}$. It is a simple Lie superalgebra.

Theorem 5.1. *Let V be a complex orthogonal vector space, $\mathfrak{m} = \mathfrak{m}_{-2} + \mathfrak{m}_{-1} = V + W$ a supertranslation algebra and $\mathfrak{g} = \sum_p \mathfrak{g}_p$ the maximal transitive prolongation of \mathfrak{m} . If $\dim V = 1$ or 2 then \mathfrak{g} is infinite-dimensional and*

- $\mathfrak{g} = K(1, N)$ if $\dim V = 1$,
- $\mathfrak{g} = K(1, N) \oplus K(1, N)$ if $\dim V = 2$.

If $\dim V \geq 3$ then \mathfrak{g} is finite-dimensional, \mathfrak{g}_0 is as in Theorem 2.3, and either $\mathfrak{g}_p = 0$ for all $p \geq 1$ or

- $\mathfrak{g} = osp(N|4)$, $\dim V = 3$ with \mathfrak{m} as in Example 4.8,
- $\mathfrak{g} = pgl(4|N)$, $\dim V = 4$ with \mathfrak{m} as in Example 4.7,
- $\mathfrak{g} = F(4)$, $\dim V = 5$ with \mathfrak{m} as in Example 4.9,

is \mathbb{Z} -graded as described in Table 3.

Proof. The cases $\dim V \geq 3$ are a direct consequence of Theorem 2.3 and Theorem 4.11. We need to prove the statement for $\dim V = 1, 2$.

If $\dim V = 1$, one has $\mathfrak{m}_{-1} = \mathbb{C}^{0|N}$ and $\mathfrak{m}_{-2} = \mathbb{C}^{1|0}$. Clearly $\mathfrak{m} \simeq K(m, n)_{-2} + K(m, n)_{-1}$, thus $\mathfrak{g} \simeq K(1, N)$.

If $\dim V = 2$, we can identify $\mathfrak{so}(2, \mathbb{C}) = \mathbb{C}$, $\mathfrak{m}_{-1} = \mathbb{C}_+^{0|N} + \mathbb{C}_-^{0|N}$, and $\mathfrak{m}_{-2} = \mathbb{C}_+^{1|0} + \mathbb{C}_-^{1|0}$, with action of $\mathfrak{so}(2, \mathbb{C})$ given by:

$$\lambda \cdot s_{\pm} = \pm \lambda s_{\pm}, \quad \lambda \cdot v_{\pm} = \pm 2\lambda v_{\pm} \quad (\lambda \in \mathfrak{so}(2, \mathbb{C}), \quad s_{\pm} \in \mathbb{C}_{\pm}^{0|N}, \quad v_{\pm} \in \mathbb{C}_{\pm}^{1|0}).$$

Then $\mathfrak{m} = \mathfrak{m}_+ \oplus \mathfrak{m}_- \simeq K(1, N)_{<0} \oplus K(1, N)_{<0}$ as a direct sum of ideals. By [26, Prop. 3.3], whose proof remains unchanged in the Lie superalgebra case, the statement follows. \square

6. COMPARISON WITH THE LIE ALGEBRA CASE

We classified the maximal transitive prolongations \mathfrak{g} of supertranslation algebras \mathfrak{m} in Theorem 5.1. If $\dim V \geq 3$ the Lie superalgebra \mathfrak{g} is finite-dimensional and either $\mathfrak{g}_p = 0$ for all $p \geq 1$ or \mathfrak{g} is isomorphic to $osp(N|4)$, $pgl(4|N)$, $F(4)$.

The analogous problem in the Lie algebra setting has been solved in [3]. In that case, the dimension of the pseudogroup of automorphisms of a manifold M endowed with an extended Poincaré structure \mathcal{D} is bounded by $\dim \mathfrak{g}$ and equality is obtained precisely when M is locally isomorphic to the maximally homogeneous model \overline{M} [35, 2].

Tanaka's results have never been proved in the superalgebra setting, although it is plausible that an appropriate version should hold true for supermanifolds. One of the problems is the lack of an established notion of “super-pseudogroup of automorphisms of a supermanifold”. Hence, the geometric implications of finite-dimensionality of \mathfrak{g} must be considered rigorously true only at a formal infinitesimal level.

Table 4 contains the list of maximal prolongations \mathfrak{g} of extended translation algebras for $\dim V \geq 3$ and with $\mathfrak{g}_1 \neq 0$ [3, Theorem 3.1]. Comparison with Table 3 reveals very clear analogies with the Lie superalgebra case for $\dim V = 3, 4$. The analogy extends to the Lie algebra F_4 and the Lie superalgebra $F(4)$, however in this case the dimension of V differs.

\mathfrak{g}	Dynkin diagram	$\dim V$	$\dim W$	N	\mathfrak{h}_0
A_ℓ $\ell \geq 4$		4	$4N$	$\ell - 3$	$\mathfrak{gl}_{\ell-3}$
C_ℓ $\ell \geq 3$		3	$2N$	$2(\ell - 2)$	$\mathfrak{sp}_{2(\ell-2)}$
F_4		7	8	1	0
E_6		8	16	1	\mathfrak{so}_2

TABLE 4.

It would be interesting to find a \mathbb{Z} -graded contragredient Lie superalgebra in correspondence with the Lie algebra E_6 of Table 4 and to look for applications to supergravity theories. We remark that it cannot be a finite-dimensional or (twisted) affine Lie superalgebra, as it follows from the classification of contragredient Lie superalgebras of finite Gelfand-Kirillov growth [24, 18]. Dynkin diagrams with shape E_6 appeared in the context of hyperbolic Kac-Moody Lie superalgebras in [6, 13] but no extensive review of their properties is known to us.

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